Existence and uniqueness of Koopman eigenfunctions near stable equilibria and limit cycles[1]

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Relevant papers and slides available

- **Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits.** MDK and Shai Revzen. Physica D (2021), arXiv:1911.11996.

- **Generic properties of Koopman eigenfunctions for stable fixed points and periodic orbits.** MDK, David Hong, and Shai Revzen. IFAC-PapersOnline (2021; MTNS conference cancelled), arXiv:2010.04008.

- **Slides available on my website**: mdkvalheim.github.io/assets/NOLTA2022.pdf

Motivation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad f(0) = 0, \quad 0 \text{ hyperbolically stable with basin } B \subset \mathbb{R}^n. \quad (1)$$

- $\psi\colon B \to \mathbb{C}$ is a **Koopman eigenfunction** if $\exists \lambda \in \mathbb{C}$ s.t.

$$\psi(x(t)) = e^{\lambda t}\psi(x_0), \quad \text{or } \dot{\psi} = \lambda\psi \text{ if } \psi \in C^1. \quad (2)$$

- Sufficiently many "independent" eigenfunctions determine an invertible change of coordinates through which (1) becomes a linear system, a drastic simplification.
- How to find eigenfunctions? If $\mu \in \mathbb{C}$ and $g\colon B \to \mathbb{C}$ is such that the limit[2]

$$g_\mu^*(x_0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(x(t))e^{-\mu t}\, dt \quad (3)$$

  exists and is not identically zero, then g_μ^* is an eigenfunction with $\lambda = \mu$ in (2).
- **Questions:** Which one? Can there be more than one possible limit (modulo scalar multiplication)? Can the limit depend sensitively on g? Other numerical issues?
- If we knew that eigenfunctions were unique, we could resolve these questions. **We will discuss uniqueness and more**, including **new convergence results** for (3).

[2] **Laplace average**: see §3 of Mauroy, Mezić, Moehlis "Isostables..." 2013. See also Mezić "Analysis..." 2012.

Principal eigenfunctions

- C^1 eigenfunctions determining a linearizing diffeomorphism must be **principal**:

$$d\psi_i(0) \neq 0.$$

- (**Fact**: if ψ_i is principal and $\dot{\psi}_i = \lambda \psi_i$, $d\psi_i(0)$ is left eigenvector of $D_0 f$ w/ e.val λ.)

- Thus, we will concentrate on existence & uniqueness of C^k principal eigenfunctions.

- In particular we will see that, under some conditions, principal eigenfunctions are uniquely determined by their derivatives at 0.

- (Later we will classify **all** C^∞ eigenfunctions under generic conditions, not just the principal ones.)

Counterexamples $\Longrightarrow$ some conditions are needed

- **Ex. 1.** Let $k \geq 2$ be an integer, $(x, y) \in \mathbb{R}^2$,

$$\dot{x} = -x, \qquad \dot{y} = -ky.$$

Both

$$\psi_1(x, y) = y \text{ and } \psi_2(x, y) = y + x^k$$

are analytic principal eigenfunctions s.t. $d\psi_1(0) = d\psi_2(0)$,

$$\dot{\psi}_i = \lambda \psi_i \text{ with } \lambda = -k.$$

$\Longrightarrow$ **nonresonance assumptions needed** (explained later).

- **Ex. 2.** Let $a > 1$ not be an integer, $(x, y) \in \mathbb{R}^2$,

$$\dot{x} = -x, \qquad \dot{y} = -ay.$$

Both

$$\psi_1(x, y) = y \text{ and } \psi_2(x, y) = y + |x|^a$$

are $C^{\lfloor a \rfloor}$ principal eigenfunctions ($\lfloor a \rfloor$ is the integer part of a) s.t. $d\psi_1(0) = d\psi_2(0)$,

$$\dot{\psi}_i = \lambda \psi_i \text{ with } \lambda = -a.$$

$\Longrightarrow$ resonance not an issue here, but **spectral spread assumptions needed** (later).

Towards C^k existence and uniqueness, step 1: reduction to discrete-time

- Henceforth assume vector field $f \in C^k$ is complete with C^k flow $(t, x) \mapsto \Phi^t(x)$.

- Can define eigenfunctions for a diffeomorphism $F\colon B \to B$: $\psi(F(x)) = e^{\lambda}\psi(x)$.

- If eigenfunctions for $F = \Phi^1$ are unique, then they are unique for f.

- If a λ-eigenfunction $\tilde{\psi}$ for $F = \Phi^1$ exists, Sternberg's trick[3] $\implies$

  $$\psi = \int_0^1 e^{-\lambda t}\tilde{\psi} \circ \Phi^t \, dt$$

  is a λ-eigenfunction for f and $d\psi(0) = d\tilde{\psi}(0)$.

- $\implies$ **suffices to consider discrete time**, i.e. prove existence & uniqueness for principal eigenfunctions of a diffeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$, $F(0) = 0$, 0 hyperbolically stable with basin B.
- Existence & uniqueness for $k < \infty$ plus bootstrapping yields existence & uniqueness for $k = \infty$, hence assume $k < \infty$ for now.

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[3] cf. Lemma 4 of Sternberg, "Local contractions and a theorem of Poincaré" (1957).

Step 2: nonresonance and solving polynomial equations

- If $\mu \in \mathbb{C}$, $k \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ eigenvalues$(D_0 F) = e^{\lambda_1}, \ldots, e^{\lambda_n}$ repeated with multiplicity, $(e^{\mu}, D_0 F)$ is k-**nonresonant** if

$$e^{\mu} \neq e^{m_1 \lambda_1} \cdots e^{m_n \lambda_n}$$

  whenever $m_1, \ldots, m_n \in \mathbb{N}_{\geq 0}$ satisfy $2 \leq \sum_i m_i < k + 1$.

- **Key fact**:[4] If $F \in C^k$ and $\exists w \in \mathbb{C}^n$ s.t. $wD_0 F = e^{\lambda} w$, k-nonresonance $\implies$ invertibility of certain linear operators on polynomials $\implies$ $\exists!$ polynomial $P \colon \mathbb{R}^n \to \mathbb{C}$ such that $P(0) = 0$, $dP(0) = w$, and

$$P \circ F = e^{\lambda} P + o(\|x\|^k), \qquad \text{and } P \text{ is } \mathbb{R}\text{-valued if } e^{\lambda} \in \mathbb{R} \text{ and } w \in \mathbb{R}^n.$$

- In other words, k-nonresonance $\implies$ can Taylor expand and solve eigenfunction equation "order by order" to produce polynomial "eigenfunction up to order k" P.
- Remains only to **find** $o(\|x\|^k)$ **remainder** $\varphi \colon \mathbb{R}^n \to \mathbb{C}$ such that $\psi = P + \varphi$ is an eigenfunction exactly.

[4] Lemma 4 of Kvalheim and Revzen (2021).

Step 3: spectral spread and contraction mapping to eliminate remainder

- **Spectral spread** $\nu(e^{\mu}, D_0F) := \min\left\{r \in \mathbb{R} \colon |e^{\mu}| \geq \left(\max_{e^{\lambda} \in \text{evals}(D_0F)} |e^{\lambda}|\right)^r\right\}$.

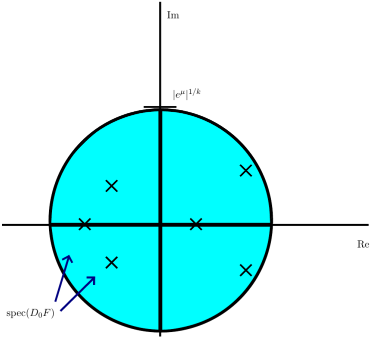

Figure: Illustration of $\nu(e^{\mu}, D_0F) < k$.

- **Key fact**: if $\nu(e^{\lambda}, D_0F) < k$, $\exists$ adapted norm $\|\cdot\|$ and $\varepsilon > 0$ s.t. with $N := B_{\varepsilon}(0)$

$$S \colon \{\varphi|_N \in C^k(N, \mathbb{C}) \colon \varphi|_N \in o(\|x\|^k)\} \circlearrowleft$$

$$S(\varphi|_N) := -P|_N + e^{-\lambda}(P|_N + \varphi|_N) \circ F$$

is a contraction mapping $\implies$ $\exists!$ $\varphi|_N$ s.t. $S(\varphi|_N) = \varphi|_N = \lim_{m\to\infty} S^m(\tilde{\varphi}|_N)$, i.e.

$$\underbrace{(P|_N + \varphi|_N)}_{\psi|_N} \circ F = e^{\lambda} \underbrace{(P|_N + \varphi|_N)}_{\psi|_N} \quad \text{and} \quad \psi|_N = \lim_{m\to\infty} e^{-\lambda m} P \circ F|_N.$$

Step 4: globalization $\implies$ discrete-time existence and uniqueness result

- Can globalize $\psi|_N \colon N \to \mathbb{C}$ to $\psi \colon B \to \mathbb{C}$ as follows: set $\psi(x) := e^{-m\lambda}\psi|_N \circ F^m(x)$ where m is large enough that $F^m(x) \in N$; can show well-defined independent of m.[5]

- **Theorem:** let $k \geq 1$, $F \in C^k(\mathbb{R}^n, \mathbb{R}^n)$, $F(0) = 0$, 0 hyperbolically stable with basin B, (e^λ, D_0F) k-nonresonant, $\nu(e^\lambda, D_0F) < k$, and $wD_0F = e^\lambda w$. Then there **exists** a **unique** C^k principal eigenfunction ψ satisfying $\psi \circ F = e^\lambda \psi$, and moreover

$$\psi = \lim_{m\to\infty} e^{-\lambda m} P \circ F \quad C^k\text{-uniformly on compacts if} \quad P \circ F = e^\lambda P + o(\|x\|^k). \tag{4}$$

- **Observation**: (4) $\implies$ Theorem hypotheses $\implies$ **convergence of Laplace average**

$$\psi = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} e^{-\lambda m} P \circ F.$$

[5] Similar techniques are used in Lan and Mezić (2013); Kvalheim, Eldering, and Revzen (2018).

Continuous-time existence and uniqueness with weaker nonresonance

- If evals$(D_0 f) = \lambda_1, \ldots, \lambda_n$ with multiplicity and $F = \Phi^1$, taking logarithm of $e^\mu \neq e^{m_1\lambda_1} \cdots e^{m_n\lambda_n}$ implies that k-nonresonance of $(e^\mu, D_0 F)$ is equivalent to

$$\mu \neq m_1\lambda_1 + \cdots m_n\lambda_n + i2\pi\ell \tag{5}$$

  for any $\ell \in \mathbb{Z}$ and any $m_1, \ldots, m_n \in \mathbb{N}_{\geq 0}$ satisfying $2 \leq \sum_i m_i < k+1$.
- By replacing $F = \Phi^1$ with $F = \Phi^\tau$ for arbitrary $\tau > 0$, (5) becomes

$$\mu \neq m_1\lambda_1 + \cdots m_n\lambda_n + i\frac{2\pi}{\tau}\ell \tag{6}$$

  which can be violated for all τ if and only if it is violated for $\ell = 0$.
- $\implies$ **Theorem**:[6] let $k \geq 1$, vector field $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, 0 hyperbolically stable with basin B, $\nu(e^\lambda, e^{D_0 f}) < k$, λ not equal to any integer linear combination of eigenvalues of $D_0 f$ with $2 \leq$ (coefficient sum) $< k+1$, and $wD_0 f = \lambda w$. Then there **exists** a **unique** C^k principal eigenfunction ψ satisfying $\psi \circ \Phi^t = e^{\lambda t}\Phi^t$ for all $t \in \mathbb{R}$, and

$$\psi = \lim_{m\to\infty} e^{-\lambda t} P \circ \Phi^t \quad C^k\text{-uniformly on compacts if} \quad P \circ \Phi^1 = e^\lambda P + o(\|x\|^k). \tag{7}$$

- **Observation**: (4) $\implies$ Theorem hypotheses $\implies$ **convergence of Laplace average**

$$\psi = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-\lambda t} P \circ \Phi^t.$$

[6] see Remark 3 of Kvalheim and Revzen (2021) or Proposition 11 of Kvalheim, Hong, and Revzen (2021).

Classification of C^∞ Koopman eigenfunctions

- **Key tool**:[7] assuming ∞-nonresonance, if $\varphi \in C^\infty(B, \mathbb{C})$ satisfies $\varphi \circ \Phi^1 = e^\lambda \varphi$ and $D_0^j \varphi = 0$ for all $j \in \mathbb{N}_{\geq 0}$, then $\varphi \equiv 0$. In particular, if $\varphi = \psi_1 - \psi_2$, $\psi_1 = \psi_2$.

- Key tool & preceding theorem can be used to prove the following.

- **Classification theorem:** let vector field $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, 0 hyperbolically stable with basin B, no eigenvalue of $D_0 f$ equal to a positive integer linear combination of the others w/ coefficient sum ≥ 2, $D_0 f$ diagonalizable over $\mathbb{C}$. Then
  - **any** Koopman λ-eigenfunction is a finite linear combination of products of n principal eigenfunctions and their complex conjugates.
  - In particular, λ is a linear combination of eigenvalues of $D_0 f$.

- This classification, and all other eigenfunction uniqueness results, were previously known only for analytic dynamics & eigenfunctions (cf. Mauroy, Mezić, Moehlis (2012)).

[7] Proposition 1 from Kvalheim and Revzen (2021).

Extension to periodic orbits

- Consider $\dot{x} = f(x)$ with $f \in C^\infty$ having a hyperbolically stable τ-periodic limit cycle with image Γ.

- Apply discrete-time versions of preceding theorems to a Poincaré map with section given by an isochron $\implies$ existence and uniqueness theorems for C^k principal eigenfunctions (those with derivatives nonvanishing on Γ).

- Corresponding classification theorem has a twist involving the unique C^∞ **asymptotic phase** eigenfunction ψ_θ satisfying[8] $\dot{\psi}_\theta = i\frac{2\pi}{\tau}\psi_\theta$.

- **Classification theorem:** let $f \in C^\infty$ and assume no Floquet multiplier is a finite product of positive integer powers of the others with power sum ≥ 2. Then
  - **any** Koopman λ-eigenfunction is a finite linear combination of products of $(n-1)$ principal eigenfunctions and ψ_θ and their complex conjugates.
  - In particular, e^λ is a product of powers of Floquet multipliers.

[8] See Mauroy and Mezić "On the use of Fourier..." (2012), Kvalheim and Revzen (2021).

Remarks on other results from Kvalheim and Revzen (2021)

- Results are given for both continuous-time and discrete-time.
- Main theorem is actually existence/uniqueness of general linearizing **semiconjugacies** (aka **factors**): maps $\psi \colon B \to \mathbb{C}^m$ s.t. $\psi \circ \Phi^t = e^{At}\psi$ with $A \in \mathbb{C}^{m\times m}$.
- Application in paper: improvements of the Sternberg linearization and Floquet normal form theorems, with uniqueness statements, without assuming diagonalizable linearized dynamics.
- Paper considers $\psi \in C^{k,\alpha}$, i.e. $\psi \in C^k$ such that $D^k\psi$ is locally Hölder continuous with exponent α. With this, results become fairly sharp (as examples in paper show).
- Stronger uniqueness-only statements in paper only require C^1 (not C^k) dynamics, but existence no longer guaranteed for merely C^1 dynamics.
- Paper discusses in detail implications for **isostables** and **isostable coordinates** from literature—these exist and are unique under much weaker conditions than needed to guarantee that a full C^k linearization exists.
- Also, see Schlosser and Korda "Sparsity structures for Koopman and Perron-Frobenius operators", SIADS (2022) for an interesting application of the uniqueness results.

Thank you for your time and attention, and thank you to the organizers Milan Korda, Alexandre Mauroy, Igor Mezić, and Yoshihiko Susuki for their kind invitation.