Large deviations, persistent homology, and Brownian conductors with negative resistance[1]

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My slides are available: seas.upenn.edu/$\sim$kvalheim/assets/cornell-2-21-2022.pdf

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[1] arXiv:2108.06431. Funding from the ARO under the SLICE MURI Program is gratefully acknowledged.

Outline

1. Flux and how to create it: flux $> 0 \iff$ tilt > 0, noise > 0
2. Toward a quantitative theory of flux: small-noise asymptotics $\leftarrow$ longest PH_0 bar
3. Understanding Brownian conductors with negative resistance
4. Some ideas behind the proof of the PH_0 flux large deviations result

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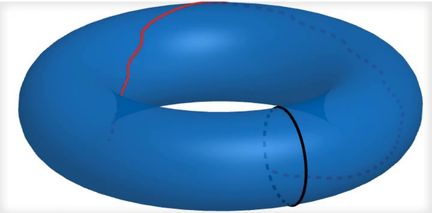
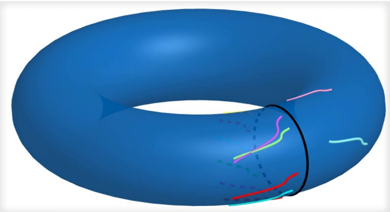
This talk is about the "flux" of a Brownian motion with drift on a manifold





What this talk is about:

- **Nondegenerate diffusion processes**[2] on smooth manifolds M ($= \mathbb{T}^2$ above)
  - with smooth infinitesimal generator $\implies$ $\exists$ metric with respect to which the infinitesimal generator $L_\epsilon = \mathbf{v}_\epsilon + \epsilon\Delta$, and the probability density ρ_t satisfies

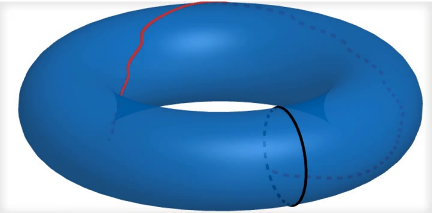
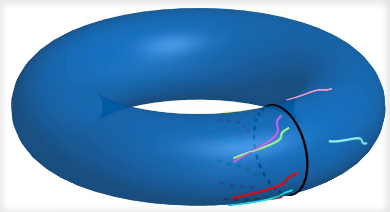
$$\partial_t \rho_t = -\nabla \cdot (\rho_t \mathbf{v}_\epsilon) + \epsilon \Delta \rho = -\nabla \cdot \underbrace{(\rho \mathbf{v}_\epsilon - \epsilon \nabla \rho)}_{J(t)=\text{probability current}} .$$

- And their **flux**: given closed hypersurface $N \subset M$ (black circle above),

$$\mathcal{F}_{\epsilon,N,t} := \int_N \langle J(t), \hat{n}_N \rangle dy.$$

[2] McKean (1969), Ikeda & Watanabe (1989), Hsu (2002).

More specifically, the steady-state flux

Nondegenerate diffusion processes in steady state:

- **Steady-state** probability **density** ρ_ϵ and **current** J_ϵ:

$$0 = -\nabla \cdot (\rho_\epsilon \mathbf{v}_\epsilon) + \epsilon \Delta \rho_\epsilon = -\nabla \cdot \underbrace{(\rho_\epsilon \mathbf{v}_\epsilon - \epsilon \nabla \rho_\epsilon)}_{J_\epsilon}.$$

- **Steady-state flux**: given closed hypersurface $N \subset M$ (black circle above),

$$\left.\mathcal{F}_{\epsilon,N} := \int_N \langle J_\epsilon, \hat{n}_N \rangle dy. \qquad \right\} \leftarrow \text{“macroscopic” definition}$$

- **Enter topology:** since $\nabla \cdot J_\epsilon = 0$, $\mathcal{F}_{\epsilon,N}$ depends only on the homology class $[N]$ (divergence theorem), and if $[\alpha] \in H^1_{\mathrm{dR}}(M)$ is Poincaré dual to $[N]$ ($\alpha = d\theta^1$ above),

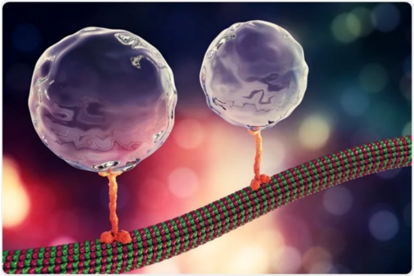
$$\left.\mathcal{F}_\epsilon([\alpha]) := \mathcal{F}_{\epsilon,N} \overset{\text{a.s}}{=} \lim_{t\to\infty} \frac{1}{t} \int_{X_{[0,t]}} \alpha. \qquad \right\} \leftarrow \text{“microscopic” definition (Manabe 1982)}$$

Some non-mathematical motivations[3]

- This class of models has been used to describe:

  Phase-locked loops, Josephson junctions, rotating dipoles in external fields, superionic conductors, charge density waves, synchronization phenomena, diffusion on crystal surfaces, particle separation by electrophoresis, biophysical processes such as intracellular transport,...

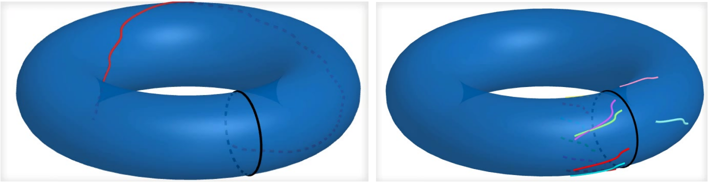
  cf. Risken (1989), Reimann et al. (2001).



Intracellular transport, kinesin motor proteins transport molecules moving across microtubules, 3D illustration. Kateryna Kon / Shutterstock

[3] Image: www.news-medical.net/news/20190715/New-insights-into-molecular-motors-could-help-treat-neurological-disorders.aspx

Steady-state flux: generalizing beyond hypersurfaces



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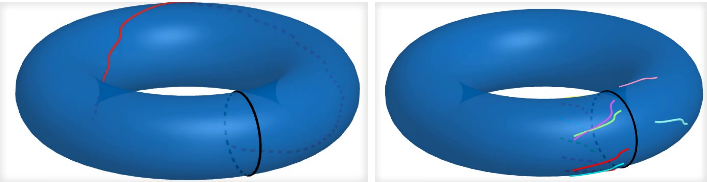
- **More generally:** steady-state **flux** is a linear $\mathcal{F}_\epsilon \colon H^1_{\mathrm{dR}}(M) \to \mathbb{R}$ (Schwartzman 1957),

$$\mathcal{F}_\epsilon([\alpha]) := \int_M \alpha(J_\epsilon) dx \overset{\text{a.s.}}{=} \lim_{t\to\infty} \frac{1}{t} \int_{X_{[0,t]}} \alpha.$$

Steady-state flux in general: a linear functional on $H^1_{\mathrm{dR}}(M)$



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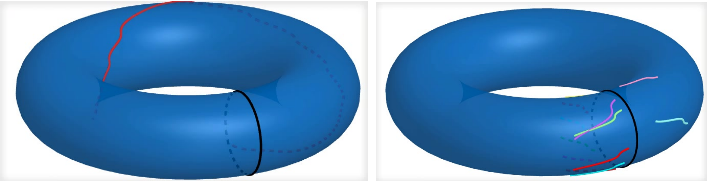
- When $[\alpha]$ is Poincaré dual to (cooriented) closed hypersurface $N \subset M$,

$$\mathcal{F}_\epsilon([\alpha]) = \mathcal{F}_{\epsilon,N}.$$

Hence $\mathcal{F}_\epsilon([\alpha])$ indeed **generalizes** $\mathcal{F}_{\epsilon,N}$.

Control Q: how can we create positive flux through a desired $[\alpha] \in H^1_{\mathrm{dR}}(M)$?



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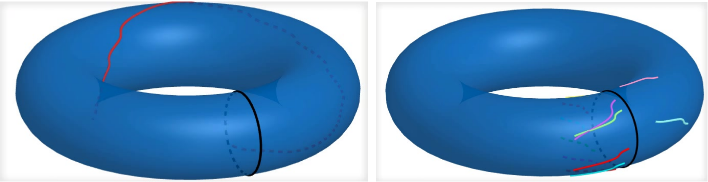
- **Control question**: how can we create a positive flux through $[\alpha] \in H^1_{\mathrm{dR}}(M)$?

Control Q answer: set $\mathbf{v}_\epsilon$ proportional to $\alpha^\sharp$



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- **Answer**: assume $\mathbf{v}_\epsilon$ is a "control force" we get to choose. Choose $\mathbf{v}_\epsilon = c\alpha^\sharp$, $c > 0$.

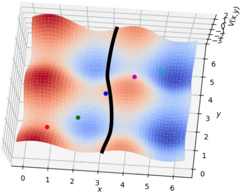
Proposition (YMB & MDK 2021): Assume $\mathbf{v}_\epsilon = \alpha^\sharp$. Then for the diffusion on M with generator $L_\epsilon = \alpha^\sharp + \epsilon\Delta$:

$$\mathcal{F}_\epsilon([\alpha]) = \int_M \frac{\|J_\epsilon\|^2}{\rho_\epsilon} dx \geq 0$$

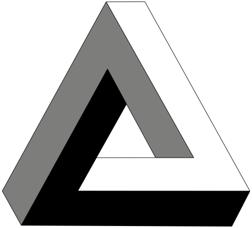
with equality $\iff$ α is exact (i.e., $\alpha^\sharp = -\nabla U$ for some $U \in C^\infty(M)$).

Ways to think about local gradients $\alpha^\sharp$ (closed one-forms α)

"Unwrap space" to obtain an honest gradient



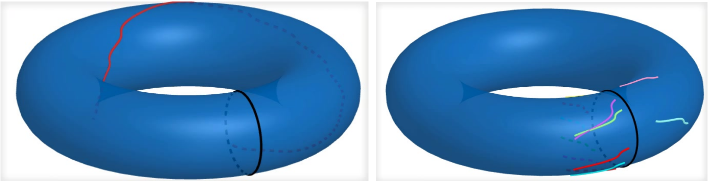
Or instead: do not unwrap, view $\alpha^\sharp$ as gradient of "impossible landscape"



Penrose tribar

Escher's "Waterfall"

Control Q answer: set $\mathbf{v}_\epsilon$ proportional to $\alpha^\sharp$



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Proof of proposition

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Proof: Recall $J_\epsilon := \rho_\epsilon \alpha^\sharp - \epsilon \nabla \rho_\epsilon$, expand $\|J_\epsilon\|^2 = \langle J_\epsilon, J_\epsilon \rangle$:

$$\frac{\|J_\epsilon\|^2}{\rho_\epsilon} = \langle \frac{\rho_\epsilon \alpha^\sharp - \nabla \rho_\epsilon}{\rho_\epsilon}, J_\epsilon \rangle = \langle \alpha^\sharp - \nabla \ln \rho_\epsilon, J_\epsilon \rangle = \alpha(J_\epsilon) - \nabla \cdot [(\ln \rho_\epsilon) J_\epsilon].$$

Since $\int_M \nabla \cdot [(\ln \rho_\epsilon) J_\epsilon] dx = 0$ (divergence theorem),

$$\int_M \frac{\|J_\epsilon\|^2}{\rho_\epsilon} = \int_M \alpha(J_\epsilon) dx =: \mathcal{F}_\epsilon([\alpha]),$$

as desired. To finish, first note $J_\epsilon \equiv 0 \implies \alpha^\sharp = \nabla(\epsilon \ln \rho_\epsilon)$. Conversely, $\alpha^\sharp = -\nabla U$ $\implies e^{-\frac{1}{\epsilon}U}(-\nabla U) - \epsilon \nabla (e^{-\frac{1}{\epsilon}U}) = e^{-\frac{1}{\epsilon}U}(-\nabla U + \nabla U) = 0. \implies \rho_\epsilon \propto e^{-\frac{1}{\epsilon}U}$. $\square$

Intuition: next slide.

When $\mathbf{v}_\epsilon = \alpha^\sharp$, flux $> 0 \iff$ tilt + noise (Below $\alpha = -dU + cdx$)

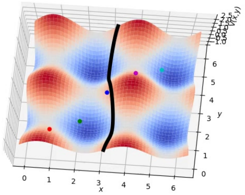
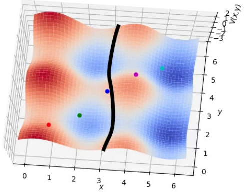


Figure: No noise, no tilt: no flux



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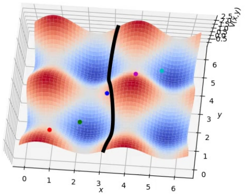

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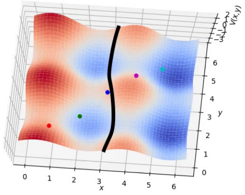


Figure: Tilt + noise: flux harvested from the noise

Other interpretation: asymptotic winding rate $> 0 \iff$ tilt + noise

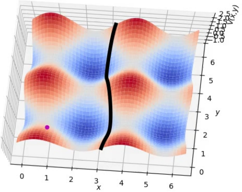


Figure: No noise, no tilt: no asymptotic winding

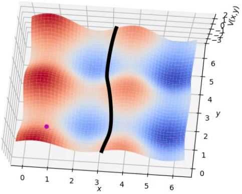
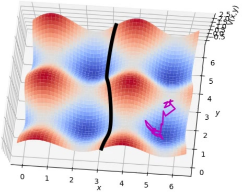


Figure: No noise: no asymptotic winding



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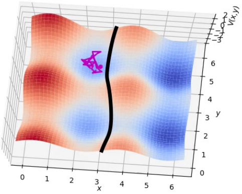

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1. Flux and how to create it: flux $> 0 \iff$ tilt > 0, noise > 0
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We know how to create a positive flux; how to control the magnitude?

How might we quantify the steady-state flux more precisely?

Why care?

Imagine technology enabled by circuits conducting Brownian particles, rather than electrons.

Basic circuit component property: resistance (inverse of conductance, mobility).

How might we determine the **resistance of a Brownian conductor**?

In the 1D case: $\exists$ closed-form solution

When $M = \mathbb{S}^1$, $\alpha^\sharp = -\partial_x U + F = -\partial_x \underbrace{(U(x) - Fx)}_{B(x)}$

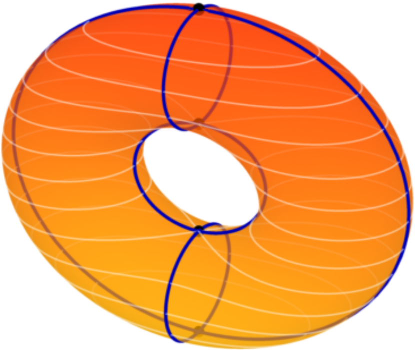
$$\mathcal{F}_\epsilon\left(\frac{[dx]}{2\pi}\right) = \frac{\epsilon(1 - e^{-\frac{F}{\epsilon}})}{\int_0^{2\pi} \int_x^{2\pi+x} e^{\frac{1}{\epsilon}(B(y)-B(x))}\, dy\, dx}.$$

- But what about when dimension$(M) > 1$? Can we "solve for" flux?
  - **Not usually**. 1D case, solving Fokker-Planck equation amounts to solving a two-point ODE BVP. $\geq$2D case: second-order linear elliptic PDE, closed-form solutions rare. (Exception: $\alpha^\sharp = -\nabla U$, but then flux $= 0$.)
- Next best thing: try to approximate flux
- **Our approach** (MDK & YMB): study the **small-noise asymptotics of flux**.

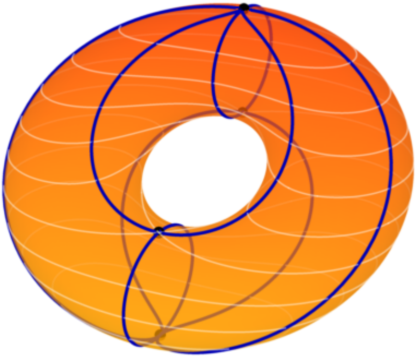
Morse-Smale functions

Assumption (U is generic)

$U \in C^\infty(M)$ has a unique global minimizer, U takes distinct values on distinct index-1 critical points, and U is Morse-Smale.



Height function $(x, y, z) \mapsto z$ on this torus **is not** Morse-Smale



Height function $(x, y, z) \mapsto z$ on this torus **is** Morse-Smale

Images: wikipedia article "Morse-Smale system"

Main result for tilted potential flux

Assumption (U is generic)

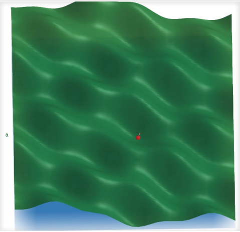
$U \in C^\infty(M)$ has a unique global minimizer, U takes distinct values on distinct index-1 critical points, and U is Morse-Smale.

- If α is C^1-close to $-dU$, there is a unique zero v_* of α close to the global min. of U.
- Assume α is closed but not exact; assume $\mathbf{v}_\epsilon \to \alpha^\sharp$ uniformly as $\epsilon \to 0$.

Theorem (YB and MDK)

If α is sufficiently C^1-close to $-dU$, the steady-state $[\alpha]$-flux of the diffusion process with generator $\mathbf{v}_\epsilon + \epsilon\Delta$ satisfies

$$\lim_{\epsilon\to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_* \qquad \left(\text{hence } \mathcal{F}_\epsilon([\alpha]) = e^{-\frac{1}{\epsilon}(h_*+o(1))}; \quad \mathcal{F}_\epsilon([\alpha]) \asymp e^{-\frac{1}{\epsilon}h_*}\right).$$



$\Omega :=$ loops $\gamma\colon [0,1] \to M$ based at v_*.
$\text{height}(\gamma) := \sup_{t\in[0,1]} \int_{\gamma|_{[0,t]}} (-\alpha)$

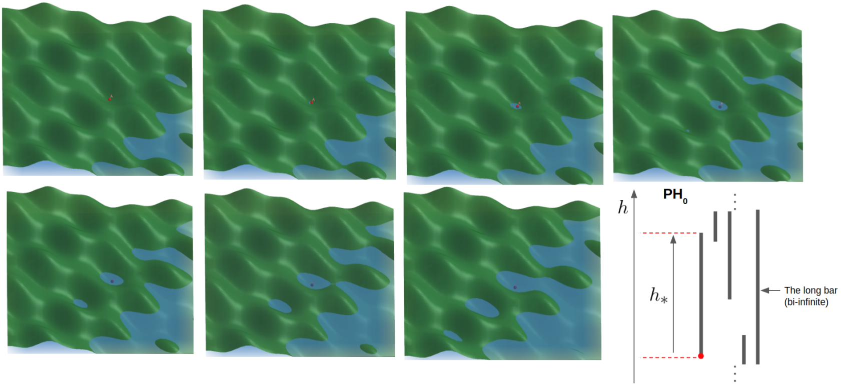
$$h_* := \inf\{\text{height}(\gamma) \colon \gamma \in \Omega \text{ and } \int_\gamma \alpha > 0\}$$

Tilted potential flux $\asymp \exp(-\frac{1}{\epsilon}(\text{critical } PH_0 \text{ bar length}))$

$$\lim_{\epsilon \to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_*$$

- "Unwrap" M: consider any cover $\pi : \tilde{M} \to M$ such that $\pi^*\alpha = -df$ is exact.
- Consider the 0th persistent homology ($\#$ components) of the filtration $\{f < h\}_{h \in \mathbb{R}}$.
- Choose a lift $\tilde{v}_* \in \pi^{-1}(v_*)$. In the zeroth persistent homology "barcode",[4]

$$h_* = \text{length(bar corresponding to } \tilde{v}_*).$$



[4] Or "merge tree". PH surveys: Ghrist (2008), Edelsbrunner and Harer (2008, 2010), Weinberger (2011).

Outline

In the 1D case: $\exists$ closed-form solution & resistance is always positive

When $M = \mathbb{S}^1$, $\alpha^\sharp = -\partial_x U + F = -\partial_x \underbrace{(U(x) - Fx)}_{B(x)}$

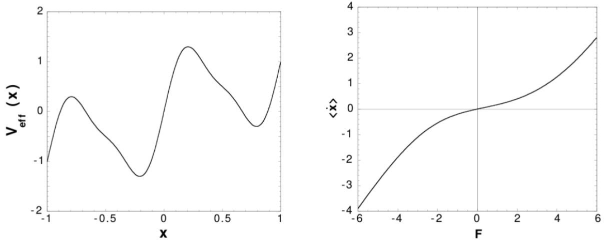
$$\mathcal{F}_\epsilon\left(\frac{[dx]}{2\pi}\right) = \frac{\epsilon\left(1 - e^{-\frac{F}{\epsilon}}\right)}{\int_0^{2\pi}\int_x^{2\pi+x} e^{\frac{1}{\epsilon}(B(y)-B(x))}\, dy\, dx}.$$

- Long known to physicists. Can compute directly to find:

$$\frac{d}{dF}\mathcal{F}_\epsilon\left(\frac{[dx]}{2\pi}\right) > 0.$$

$\implies$ **positive "resistance"**/"conductance" (greater force $\to$ greater flux/current).

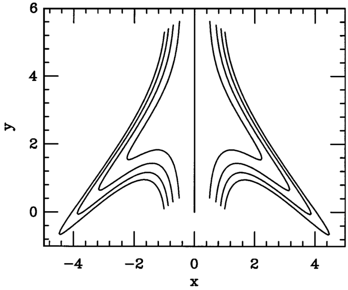
P. Reimann / Physics Reports 361 (2002) 57–265

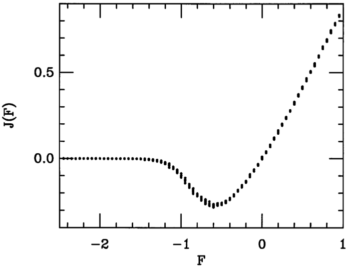


- At least, in the one-dimensional case...

What about higher-dimensional systems? Negative resistance discovered

- Cecchi and Magnasco (1996) considered the following "herringbone" potential on $\mathbb{R}^2$; demonstrated negative resistance when tilted in the downward y-direction.





- C & M's analysis: heuristic and numerical considerations.
- **Question:** does our PH_0 flux large deviations result quantify flux well enough to rigorously analyze/synthesize negative resistance?

Yes: negative resistance corollary to PH_0 result, example

$$\lim_{\epsilon \to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_*$$

Let $U \in C^\infty(M)$ satisfy the genericity assumption. Let $\alpha = -dU + c\beta$, where β is a closed but not exact one-form and $c > 0$.

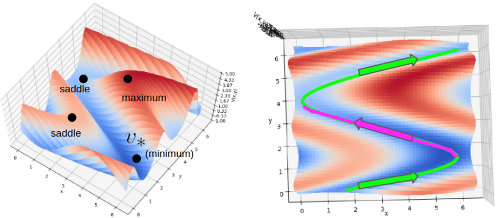
Corollary (YB and MDK)

Assume that $c \mapsto h_*(c)$ is strictly increasing on some nonempty interval $(0, c_0)$. Then for all $c_1 < c_2$ belonging to $(0, c_0)$ and all sufficiently small $\epsilon > 0$,

$$\mathcal{F}_{\epsilon,c_2}([\beta]) < \mathcal{F}_{\epsilon,c_1}([\beta]);$$

there is **negative resistance**.

Example: when small force ("tilt") $F = ce_1$ is added, $h_*(c) =$ height difference of ends of pink segment; $\frac{d}{dc}h_*(c) > 0$; $\implies$ negative resistance in the x-direction.



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Proof idea: enter Freidlin-Wentzell[5]

- Freidlin-Wentzell large deviations theory suggests we can discretize the problem and compute the small-noise flux asymptotics from an associated Markov chain (MC) on the set $V \subset M$ of attracting zeros of $\alpha^\sharp$.

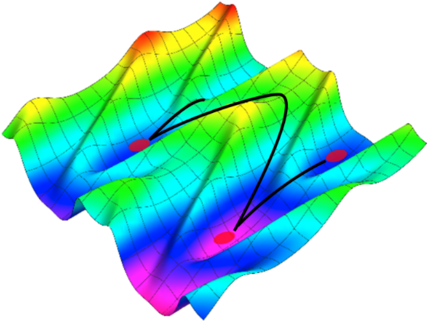


Figure: by Yuliy Baryshnikov.

- However: to detect flux, our MC needs to detect wrapping of trajectories around the manifold $\implies$ need a **finite set of vertices** V, but an **infinite set of directed edges** (we take one edge for each path homotopy class)...

[5] M I Freidlin and A D Wentzell, Random perturbations of dynamical systems, 3rd ed., 2012.

Path-homotopical refinement of Freidlin-Wentzell theory

- Given $T > 0$, the FW **action functional** $\mathcal{S}_T : C([0,T], M) \to [0, +\infty]$ is defined by

$$\mathcal{S}_T(\varphi) := \frac{1}{4} \int_0^T \|\dot{\varphi}(t) - \mathbf{v}(\varphi(t))\|^2 \, dt$$

  if φ is absolutely continuous and $\mathcal{S}_T(\varphi) := +\infty$ otherwise. Here $\mathbf{v} = \alpha^\sharp$.

- **Refined quasipotential** (YB, MDK): given **path homotopy class** $e \in \overbrace{\Pi(M)}^{\text{f. groupoid}}$,

$$Q_{\mathbf{v}}(e) := \inf\{\mathcal{S}_T(\varphi) \mid T > 0, [\varphi] = e\}.$$

- We already understand the case $\dim(M) = 1$, so henceforth $\dim(M) \geq 2$ (otherwise, special modification of $Q_{\mathbf{v}}$ needed in the 1D case).

An exact graph-theoretic expression for flux

After technicalities to construct discrete-time, continuous-space Markov chain, derive

Lemma 9. Let α be a closed one-form on M. Given $e \in E_\Pi$, define

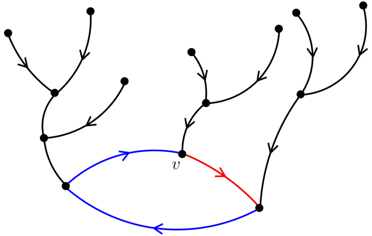
$$\bar{P}_{\kappa_1}(e) := \frac{1}{\nu^\epsilon(\partial g_{\mathfrak{s}(e)})} \int_{\partial g_{\mathfrak{s}(e)}} \nu^\epsilon(dy) P(y, e) \quad \text{and} \quad \alpha(e) := \int_e \alpha.$$

Then the following integral and sum are absolutely convergent and:

$$\mathcal{F}_\epsilon([\alpha]) = \left(\int_{\partial g} \nu^\epsilon(dy) \mathbb{E}^\epsilon_y[\tau_1] \right)^{-1} \sum_{e \in E_\Pi} \nu^\epsilon(\partial g_{\mathfrak{s}(e)}) \bar{P}_{\kappa_1}(e) \alpha(e). \tag{59}$$

Next, use Markov chain tree theorem to derive (here $\bar{\pi}(E) = \prod_{e \in E} \bar{P}_{\kappa_1}(e)$):

$$\mathcal{F}_\epsilon([\alpha]) = \left(\int_{\partial g} \nu^\epsilon(dy) \mathbb{E}^\epsilon_y[\tau_1] \right)^{-1} \left(\sum_{E \in \mathsf{RST}(\Gamma_\Pi)} \bar{\pi}(E) \right)^{-1} \sum_{E \in \mathsf{CRST}(\Gamma_\Pi)} \bar{\pi}(E) \alpha(\mathsf{cycle}(E)).$$



Tree formula, RSTs, CRSTs: Pitman & Tang (2018).

Basic lemma

Lemma (cf. Freidlin & Wentzell (2012), p. 100). Let $\mathbf{v}$ be a continuous vector field on a Riemannian manifold M. Let $\varphi \in C([T_1, T_2], M)$ satisfy $\mathcal{S}(\varphi) < \infty$. Then

$$\mathcal{S}(\varphi) = \frac{1}{4}\int_{T_1}^{T_2} \|\dot{\varphi} + \mathbf{v}(\varphi)\|^2 dt - \int_{T_1}^{T_2} \langle \dot{\varphi}, \mathbf{v}(\varphi)\rangle dt. \tag{1}$$

Proof. Let $\mathbf{w}$ be an arbitrary continuous vector field on M. By expanding both sides below,

$$\|\dot{\varphi} - \mathbf{v}(\varphi)\|^2 = \|\dot{\varphi} - \mathbf{w}(\varphi)\|^2 + \|\mathbf{v}(\varphi)\|^2 - \|\mathbf{w}(\varphi)\|^2 + 2\langle \dot{\varphi}, \mathbf{w}(\varphi) - \mathbf{v}(\varphi)\rangle.$$

Thus,

$$\begin{aligned}
\mathcal{S}(\varphi) &= \frac{1}{4}\int_{T_1}^{T_2} \|\dot{\varphi} - \mathbf{v}(\varphi)\|^2 dt \\
&= \frac{1}{4}\int_{T_1}^{T_2} \|\dot{\varphi} - \mathbf{w}(\varphi)\|^2 dt + \frac{1}{4}\int_{T_1}^{T_2} \|\mathbf{v}(\varphi)\|^2 - \|\mathbf{w}(\varphi)\|^2 dt + \frac{1}{2}\int_{T_1}^{T_2} \langle \dot{\varphi}, \mathbf{w}(\varphi) - \mathbf{v}(\varphi)\rangle dt.
\end{aligned}$$

Taking $\mathbf{w} = -\mathbf{v}$ yields (1).

Basic lemma gets parlayed into technical workhorse #1:

Proposition (YB and MDK)

Under the hypotheses of the main theorem (with $\mathbf{v} = \alpha^\sharp$),

$$Q_{\mathbf{v}}(e) = 0 \quad \Longleftrightarrow \quad \text{e contains a piecewise } \mathbf{v}\text{-integral curve}$$

and

$$Q_{\mathbf{v}}(e) = \int_e (-\alpha) \quad \Longleftrightarrow \quad \text{e contains a piecewise } (-\mathbf{v})\text{-integral curve.}$$

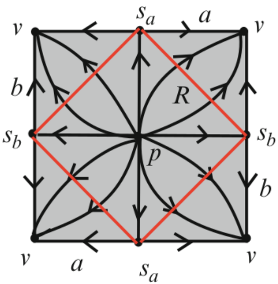


Figure: Points on red diamond index the piecewise integral curves $p \rightsquigarrow v$ in a Morse-Smale flow on $\mathbb{T}^2$. From "An Invitation to Morse theory" by Nicolaescu, 2nd ed.

Technical workhorse #1 + exact flux expression $\rightsquigarrow$ graph-theoretic LD

$$\mathcal{F}_\epsilon([\alpha]) = \left(\int_{\partial g} \nu^\epsilon(dy)\mathbb{E}_y^\epsilon[\tau_1]\right)^{-1} \left(\sum_{E\in\mathsf{RST}(\Gamma_\Pi)} \bar{\pi}(E)\right)^{-1} \sum_{E\in\mathsf{CRST}(\Gamma_\Pi)} \bar{\pi}(E)\alpha(\mathsf{cycle}(E)).$$

$M \supset g = \bigcup$ vicinities of $\alpha^\sharp$ stable zeros; τ_1 = transit. time; ν^ϵ = MC invariant measure.

Previous proposition enables estimates on the transition time and probabilities between vicinities of attracting zeros of $\alpha^\sharp$. Details+above $\implies$

Theorem. The following minima exist; assume they satisfy the inequality

$$\left(\min_{\substack{E\in\mathsf{CRST}(\Gamma_\Pi)\\ \alpha(\mathsf{cycle}(E))>0}} \sum_{e\in E} Q_{\mathsf{v}}(e)\right) < \left(\min_{\substack{E\in\mathsf{CRST}(\Gamma_\Pi)\\ \alpha(\mathsf{cycle}(E))<0}} \sum_{e\in E} Q_{\mathsf{v}}(e)\right). \tag{2}$$

Then $\mathcal{F}_\epsilon([\alpha]) > 0$ for all sufficiently small $\epsilon > 0$, and

$$\lim_{\epsilon\to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = \left(\min_{\substack{E\in\mathsf{CRST}(\Gamma_\Pi)\\ \alpha(\mathsf{cycle}(E))>0}} \sum_{e\in E} Q_{\mathsf{v}}(e)\right) - \left(\min_{E\in\mathsf{RST}(\Gamma_\Pi)} \sum_{e\in E} Q_{\mathsf{v}}(e)\right). \tag{3}$$

Downside: evaluating $Q_{\mathsf{v}}(\,\cdot\,)$ is a generally-difficult calculus of variations problem.

Toward getting around the downside: technical workhorse # 2

Proposition (YMB and MDK): continuity of Q at Morse-Smale vector fields.
Let $\mathbf{v}_0 \in \mathfrak{X}^1(M)$ be a Morse-Smale vector field without nonstationary periodic orbits on closed Riemannian manifold M. Then for any $e_0 \in \Pi(M)$, the map

$$(\mathbf{v}, e) \in \mathfrak{X}^1(M) \times \Pi(M) \mapsto Q_{\mathbf{v}}(e) \in [0, +\infty) \quad \text{is continuous at } (\mathbf{v}_0, e_0).$$

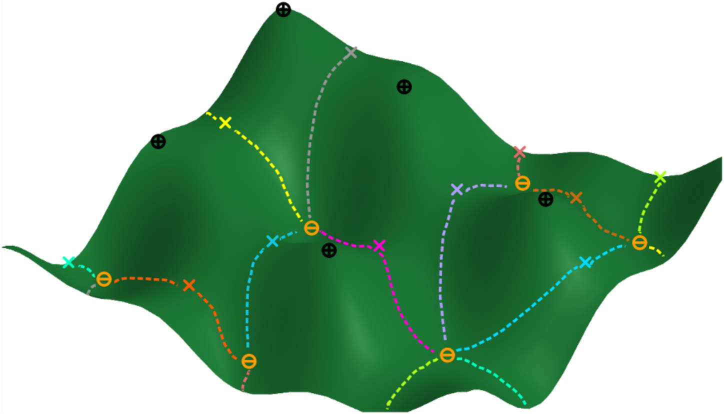
Here $\pi \colon \tilde{M} \to M$ is the universal cover and the deck group $\mathrm{Aut}(\pi)$ acts diagonally on $\tilde{M} \times \tilde{M}$, equipping $\Pi(M) \approx (\tilde{M} \times \tilde{M})/\mathrm{Aut}(\pi)$ with the topology and structure of a smooth manifold of dimension $2\dim(M)$.

Remark. I do not know if Q is continuous at $(\mathbf{v}_0, e_0)$ for general $\mathbf{v}_0 \in \mathfrak{X}^1(M)$.
(open question?)

Technical workhorse # 2 enables the second step in the following argument

Vague sketch of rest of the proof of the PH_0 "longest bar" flux small-noise LD theorem:

1. If $\mathbf{v} = -\nabla U$, the $Q_{\mathbf{v}}$-optimal rooted spanning trees all coincide as undirected graphs as the root ranges over all vertices. Moreover,
   - the RST minima with respect to $Q_{\mathbf{v}}$ in the path-homotopical graph coincide with minima with respect to saddle-minus-minimum cost in the Morse 1-skeleton, and
   - the $Q_{\mathbf{v}}$-cost of the optimal tree rooted at v minus that of the optimal tree rooted at w is $U(v) - U(w)$.



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   - the $Q_{\mathbf{v}}$-cost of the optimal tree rooted at v minus that of the optimal tree rooted at w is $U(v) - U(w)$.
2. Using the continuity of Q at Morse-Smale $-\nabla U$, other Q estimates, discreteness of the space of rooted spanning trees:
   - Can show that, if $\mathbf{v} = \alpha^\sharp$ is sufficiently C^1-close to generic $U \in C^\infty(M)$, then the RST $Q_{\mathbf{v}}$-minimizers have the same property but with U replaced by a potential for α defined on a certain spanning tree in the Morse 1-skeleton.
   - Similar analysis for the CRST minimizers.
3. This allows the graph-theoretic expressions involving quasipotentials to be simplified to Morse-theoretic expressions, yielding the longest PH_0 bar theorem.

Remark: the "small tilt" (C^1-close to generic gradient) hypothesis is crucial

- Not enough for $\alpha^\sharp$ to be Morse-Smale without periodic orbits.
- When α is sufficiently C^1-close to $-dU$ for generic $U \in C^\infty(M)$, trajectories circulate around one "main highway" with high probability.
- When tilt is large, trajectories can "fall off" main highway, take exponentially long times to "climb back on".

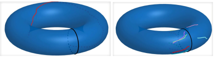
- Explicit $\mathbb{T}^2$ counterexamples (YB, MDK) show our hypotheses are fairly sharp.

Outline

1. Flux and how to create it: flux $> 0 \iff$ tilt > 0, noise > 0
2. Toward a quantitative theory of flux: small-noise asymptotics $\leftarrow$ longest PH_0 bar
3. Understanding Brownian conductors with negative resistance
4. Some ideas behind the proof of the PH_0 flux large deviations result

Flux in tilted potential systems: negative resistance and persistence[6]

Control Q answer: set $\mathbf{v}_\epsilon$ proportional to $\alpha^\sharp$



- When $[\alpha]$ is Poincaré dual to (cooriented) closed hypersurface $N \subset M$,

$$\mathcal{F}_\epsilon([\alpha]) = \mathcal{F}_{\epsilon,N},$$

  hence $\mathcal{F}_\epsilon([\alpha])$ indeed **generalizes** $\mathcal{F}_{\epsilon,N}$.
- **Control question**: how can we create a positive flux through $[\alpha] \in H^1_{\mathrm{dR}}(M)$?
- **Answer**: assume $\mathbf{v}_\epsilon$ is a "control force" we get to choose. Choose $\mathbf{v}_\epsilon = c\alpha^\sharp$, $c > 0$.

Proposition (YMB & MDK 2021): Assume $\mathbf{v}_\epsilon = \alpha^\sharp$. Then for the diffusion on M with generator $L_\epsilon = \alpha^\sharp + \epsilon\Delta$:

$$\mathcal{F}_\epsilon([\alpha]) = \int_M \frac{\|J_\epsilon\|^2}{\rho_\epsilon} dx \geq 0$$

with equality $\iff$ α is exact (i.e., $\alpha^\sharp = -\nabla U$ for some $U \in C^\infty(M)$).

When $\mathbf{v}_\epsilon = \alpha^\sharp$, flux > 0 $\iff$ tilt + noise (Below $\alpha = -dU + cdx$)

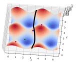


Figure: No noise, no tilt: no flux

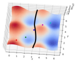
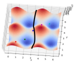


Figure: No noise: no flux



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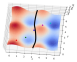


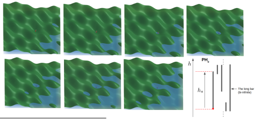
Figure: Tilt + noise: flux harvested from the noise

Tilted potential flux $\asymp \exp(-\frac{1}{\epsilon}(\text{critical PH}_0 \text{ bar length}))$

$$\lim_{\epsilon \to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_*$$

- "Unwrap" M: consider any cover $\pi\colon \tilde{M} \to M$ such that $\pi^*\alpha = -df$ is exact.
- Consider the 0th persistent homology (# components) of the filtration $\{f < h\}_{h\in\mathbb{R}}$.
- Choose a lift $\tilde{v}_* \in \pi^{-1}(v_*)$. In the zeroth persistent homology "barcode",[4]

$$h_* = \text{length}(\text{bar corresponding to } \tilde{v}_*).$$



[4] Or "merge tree". PH surveys: Ghrist (2008), Edelsbrunner and Harer (2008, 2010), Weinberger (2011).

Yes: negative resistance corollary to PH$_0$ result, example

$$\lim_{\epsilon \to 0}(-\epsilon \ln \mathcal{F}_\epsilon([\alpha])) = h_*$$

Let $U \in C^\infty(M)$ satisfy the genericity assumption. Let $\alpha = -dU + c\beta$, where β is a closed but not exact one-form and $c > 0$.

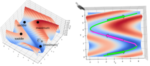
Corollary (YB and MDK)

Assume that $c \mapsto h_*(c)$ is strictly increasing on some nonempty interval $(0, c_0)$. Then for all $c_1 < c_2$ belonging to $(0, c_0)$ and all sufficiently small $\epsilon > 0$,

$$\mathcal{F}_{\epsilon,c_2}([\beta]) < \mathcal{F}_{\epsilon,c_1}([\beta]);$$

there is **negative resistance**.

Example: when small force ("tilt") $F = ce_1$ is added, $h_*(c) =$ height difference of ends of pink segment; $\frac{d}{dc}h_*(c) > 0$; $\implies$ negative resistance in the x-direction.



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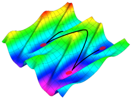

Figure: by Yuliy Baryshnikov.

- However: to detect flux, our MC needs to detect wrapping of trajectories around the manifold $\implies$ need a **finite set of vertices** V, but an **infinite set of directed edges** (we take one edge for each path homotopy class)...

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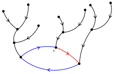

Tree formula, RSTs, CRSTs: Pitman & Tang (2018).

THANK YOU

[6] Matthew Kvalheim (kvalheim@seas.upenn.edu) and Yuliy Baryshnikov. arXiv:2108.06431

"Large tilt" counterexamples