Linearizability of dynamical systems by embeddings

Matthew D. Kvalheim (joint with Philip Arathoon)

Department of Mathematics and Statistics
University of Maryland, Baltimore County

kvalheim@umbc.edu

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"Applied Koopmanism" assumes globally linearizable dynamics

Which dynamical systems are globally linearizable?

1-parameter subgroups of torus actions with asymptotic phase

“Applied Koopmanism”

“*A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.*”

— Brunton, Budišić, Kaiser, and Kutz. “Modern Koopman Theory for Dynamical Systems.” SIAM Review, 64.2 (2022)

More formally, they seek *embeddings* of *nonlinear* dynamical systems into *linear* ones as invariant subsets, so that existing theoretical and algorithmic linear tools can be utilized.

Linearizing embeddings

Let f be a locally Lipschitz vector field on a manifold M. Consider

$$\dot{x} = \frac{d}{dt}x = f(x),$$

assume this ODE's solutions $x(t) = \Phi^t(x_0)$ are defined for all time.

$F\colon M \to \mathbb{R}^n$ is a **topological embedding** if F is a one-to-one continuous map with a continuous inverse $F^{-1}\colon F(M) \to M$, and is a **smooth embedding** if additionally F, F^{-1} are smooth.

Such an embedding F is **linearizing** if $F \circ \Phi^t = e^{Bt} \circ F$ for some $n \times n$ matrix B. In the smooth case, $y = F(x)$ satisfies $\dot{y} = By$.

Fundamental question: when is (M, Φ) linearizable in this sense?

When is a dynamical system (M, Φ) globally linearizable?

- Not when M is connected, forward Φ-trajectories are precompact, and Φ has a countable number ≥ 2 of omega limit sets (Liu, Ozay, Sontag 2023).
- Not when M is connected and Φ has a non-global compact attractor $A \neq \varnothing$, since its open basin of attraction would also be closed (by the Jordan normal form theorem), hence empty.

Thus, we study linearizability of the restriction (S, Φ) of Φ to

1. compact invariant sets S, and
2. basins S of compact attractors A.

For these 2 cases we obtain **necessary and sufficient conditions for global linearizability** of (S, Φ) by an embedding, for the 2 cases of topological and smooth embeddings (4 cases total).[1]

[1]MDK and P. Arathoon, *Linearizability of flows by embeddings (2023).*

Torus preliminaries

The n-torus $T = T^n$ is Lie group isomorphic to $(\mathbb{R}/\mathbb{Z})^n$, vectors with n real entries but with addition defined elementwise modulo 1.

A **torus action** on S is a map $\Theta \colon T \times S \to S$ satisfying $\Theta^{\tau_1+\tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

The flow (S, Φ) is a **1-parameter subgroup of a torus action** if $\Phi^t = \Theta^{\omega t \bmod 1}$ for some torus action Θ on S, $\omega \in \mathbb{R}^n \cong \mathrm{Lie}(T^n)$.

The linearizability theorem, case 1: compact, smooth

Observation: If (S, Φ) is linearizable with S compact, the Jordan normal form theorem implies (S, Φ) embeds into the flow on $\mathbb{C}^n$ of a diagonal imaginary matrix, so (S, Φ) is a 1-parameter subgroup of restriction of standard torus action of T^n on $\mathbb{C}^n$ to a subtorus.

This gives one implication below; the Mostow-Palais equivariant embedding theorem gives the other.

Theorem (MDK and P. Arathoon). If S is a compact embedded submanifold, (S, Φ) is linearizable by a smooth embedding $\iff$ (S, Φ) is a 1-parameter subgroup of a smooth torus action.

We use this theorem to construct examples of smoothly linearizable (S, Φ) having isolated equilibria with e.g. $S =$ a sphere, torus, Klein bottle. On the other hand, regarding *non*linearizability...

Topological implications for case 1 (compact, smooth)

If (S, Φ) is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an $n \times n$ skew matrix B_e and a system of local coordinates on a neighborhood of each equilibrium $e \in S$ such that $\Phi^t \approx e^{B_e t}$. Hence if e is isolated then B_e is invertible, $n = \dim S$ is even, and the Hopf index of e is $+1$.

Corollary (MDK and PA). If S is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then (S, Φ) cannot be linearized by a smooth embedding.

Corollary (MDK and PA).[2] If S is a compact submanifold containing at most finitely many equilibria such that (S, Φ) is linearizable by a smooth embedding, $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$.

[2] Apply the Poincaré-Hopf theorem.

The linearizability theorem, case 2: compact, continuous

The theorem for case 2 is similar for case 1, but an additional assumption is needed to rule out a pathology not possible in case 1.

A torus action has **finitely many orbit types** if there are only finitely many subgroups $H \subset T$ such that $H = \{\tau \in T : \Theta^\tau(s) = s\}$ is the fixed point set of some $s \in S$.

Theorem (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff$ (S, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

Another point of view: quasiperiodic pinched torus families

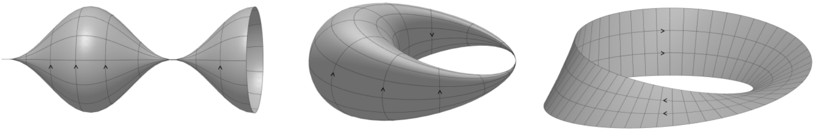

Figure: examples of quasiperiodic pinched torus families

Proposition (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff$ (S, Φ) is a quasiperiodic pinched torus family.

Definition. P is a **pinched torus family** if there are $m, n \in \mathbb{N}$, closed subsets $C_1, \ldots, C_n \subset B \subset T^m$, and a continuous group homomorphism $F\colon T^n \to T^m$ such that P is the quotient of $F^{-1}(B)$ by collapsing the j-th $(\mathbb{R}/\mathbb{Z})$-factor of $F^{-1}(C_j) \subset T^n$ for all j. A pinched torus family P is **quasiperiodic** if it is equipped with the induced flow generated by any $\omega \in \mathbb{R}^n$ with $\mathsf{T}F(\omega) = 0$.

The linearizability theorem, case 3: basin, continuous

If S is the basin of an asymptotically stable compact set $A \subset S$, A has continuous (smooth) **asymptotic phase**[3] if there is a continuous (smooth) **asymptotic phase map** $P\colon S \to A$, i.e.,

$$P|_A = \mathrm{id}_A, \qquad P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

Theorem (MDK and PA). (S, Φ) is linearizable by a topological embedding $\iff$ A has continuous asymptotic phase and (A, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

Example. The basin of an asymptotically stable limit cycle is linearizable by a topological embedding $\iff$ the cycle has continuous asymptotic phase. This is not always the case, but it is the case if $\Phi \in C^1$ and the cycle is hyperbolic.

[3] This notion has roots in oscillator theory and more generally NHIM theory.

The linearizability theorem, case 4: basin, smooth

Theorem (MDK and PA). (S, Φ) is linearizable by a smooth embedding $\iff$ A is an embedded submanifold with smooth asymptotic phase, (A, Φ) is a 1-parameter subgroup of a smooth torus action, and for some open $U \supset A$, (U, Φ) embeds in a reducible linear flow covering Φ on some vector bundle over A.

When does the final condition hold? Classical linearization theorems and recent linearizing semiconjugacy theorems (MDK and Revzen, 2023) give answers in the special cases that A is an equilibrium or periodic orbit, and some things are known if A is quasiperiodic, but the general case seems to be an open problem.

A necessary condition for A to satisfy all conditions of the theorem is that A be a (eventually relatively ∞-)**normally hyperbolic invariant manifold**. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

Thank you for your time and attention.

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