# ASYMPTOTIC STABILITY AND DIFFERENTIAL TOPOLOGY

MATTHEW D. KVALHEIM

*To Stephen Smale for his 95th birthday*

Abstract. We survey classical results and recent developments that show how differential topology shapes our understanding of asymptotic stability of equilibria in dynamical systems.

## 1. Introduction

1.1. **Motivation and scope.** This paper surveys aspects of asymptotic stability of dynamical systems where differential topology plays a crucial role. The author is very happy to dedicate this paper to Stephen Smale, whose foundational ideas and results in differential topology are central to Sections 4 and 5.

Roughly speaking, an equilibrium point of a dynamical system is asymptotically stable if trajectories starting near the equilibrium remain near and converge to the equilibrium asymptotically in time [BS70, HS74]. Thus, asymptotic stability models robust steady-state behavior of a type desirable for many real-world systems.

A number of fundamental questions concerning asymptotic stability have been answered using methods of differential topology, the study of global properties of smooth manifolds and maps [Abr63, Mil97, Hir94, Kos93, Wal16]. The goal of this survey is to explain such results and relationships between them, which are perhaps not so widely known. Along the way, we give or sketch a number of proofs to illustrate the role of differential topology in stability theory. To the author's knowledge, Proposition 5 is new.

This paper is not a comprehensive review of stability theory, but a biased tour of interconnected results showing how differential topology shapes it. We do not discuss exponential stability [Mas56], time-varying [Mas56, Wil69, Cor92] or discontinuous [CLSS97, CLS98] systems, dynamic feedback [BK25], or asymptotic stability of invariant sets more general than equilibria [Wil67, Wil69, BS70, Man07, Kva23].

1.2. **Outline of the paper.** In Section 2, we define stability and asymptotic stability of equilibria, and we discuss Lyapunov's methods for determining whether an equilibrium has one of these properties [Lya92]. We then recall the converse Lyapunov function theorem of Kurzweil and Massera [Kur56, Mas56]. This provides a converse to one of Lyapunov's methods and a bridge between asymptotic stability and differential topology. Using this and the Poincaré–Hopf theorem [Poi85, Hop27], we give a short proof of topological necessary conditions for asymptotic stability due to Bobylev and Krasnosel'skiĭ [BK74] and Brockett (see also [Zab89, KK22]).

In Section 3, following Wilson, we prove that the basin of attraction of an asymptotically stable equilibrium is always diffeomorphic to Euclidean space [Wil67]. This statement implies well-known results [BB00], and is an important tool for later sections. The proof of this result relies on a theorem of Brown and Stallings [Bro61, Sta62], whose proof in turn relies on the disc theorem of Palais and Cerf [Pal60, Cer61]. Since the disc theorem appears again in Section 5, we illustrate its use by giving Milnor's proof of the Brown–Stallings theorem [Mil64].

In Section 4, we study the topology of level and sublevel sets of Lyapunov functions following Wilson's methods [Wil67]. The main result asserts that sublevel sets of smooth proper strict Lyapunov functions are always homeomorphic to discs, and diffeomorphic to discs in most dimensions; a similar statement holds for level sets and spheres. This is a combination of earlier observations [Wil67, GSW99, Byr08, Jon26], and the main work in the proof is done by differential topology. We rely on Smale's proof of the topological Poincaré conjecture for smooth manifolds in dimensions greater than 4 and his $h$-cobordism theorem [Sma61, Sma62], Freedman's proof of the topological Poincaré conjecture in dimension 4 [Fre82], the generalized Schoenflies theorem of Mazur and Brown [Maz59, Bro60], and Perelman's resolution of the Poincaré conjecture in dimension 3 [Per02, Per03b, Per03a].

As an application, we prove an extension of the Hartman–Grobman linearization theorem to nonhyperbolic but asymptotically stable equilibria as in [KS25] (see also [Col65, Col66, Wil78, GSW99]). Using this result together with the stable homeomorphism theorem [Kir69, Qui82] (see also [Edw84]), we can then partially address a question asked by Conley about "continuation" of isolated invariant sets of dynamical systems [Con78] in Proposition 5, an extension of a recent theorem of Jongeneel [Jon26].

Having studied various spaces associated with asymptotically stable equilibria, in Section 5 we turn instead to the study of the function space of all $C^k$ vector fields having a globally asymptotically stable equilibrium, for $1 \leq k \leq \infty$. We equip this space with the compact-open $C^k$ topology, the natural choice for our applications [Hir94, Section 2.1]. Following [Kva25a], we sketch the proof that the space of all such vector fields has the same weak homotopy type as the nonlinear Grassmannian of $n$-discs in $\mathbb{R}^n$ [GBV14]. From this it follows that the space of $C^k$ asymptotically stable vector fields on $\mathbb{R}^n$ is path-connected and simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$.[1] We give a proof of this implication, relying heavily on Smale's result that the space $\mathrm{Diff}_\partial(D^2)$ of diffeomorphisms of the 2-disc restricting to the identity on $\partial D^2$ is contractible [Sma59], Hatcher's proof of the analogous Smale conjecture for $D^3$ [Hat83], and Cerf's proof of the pseudo-isotopy theorem, in which he developed one-parameter versions of Smale's handle-theoretic methods [Cer70].

Following [Kva25a], we then describe applications including parametric asymptotically stable "boundary value problems" relevant to feedback stabilization [Son98, Cor07, Zab20, JM23]. We then return to Conley's question and mention a smooth result complementing the topological results of Jongeneel [Jon26] and Proposition 5, as well as other smooth continuation results of Reineck [Rei91, Rei92] and recently Mischaikow and Parikh [MP26]. Finally, we give parametric extensions of the results in Section 4 concerning level and sublevel sets of Lyapunov functions, and Hartman–Grobman linearization.

[1] It is interesting to note that, for any $1 \leq k \leq \infty$, the space of $C^k$ asymptotically stable vector fields on $\mathbb{R}^5$ is path-connected if and only if the 4-dimensional smooth Poincaré conjecture is true (see Remark 10), and a similar remark holds for Proposition 4 (see [KS25, Proposition 1]).

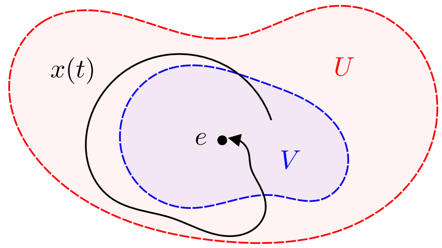


FIGURE 1. Asymptotic stability.

## 2. Asymptotic stability and stability tests

Let $M$ be a smooth ($C^\infty$) manifold and $f$ a vector field on $M$ defining a system

$$\dot{x} = \frac{d}{dt}x = f(x) \tag{1}$$

of nonlinear ordinary differential equations. We assume that $f$ is continuous and **uniquely integrable**, meaning (1) has a unique maximal solution for each choice of initial condition $x(0)$. (This is automatic if $f$ is $C^1$ or, more generally, locally Lipschitz.) Let $e \in M$ be an **equilibrium** point or **zero**, meaning $f(e) = 0$.

The equilibrium $e$ is **stable** if, for every neighborhood $U$ of $e$, there is a neighborhood $V \subset U$ of $e$ such that every solution $x(t)$ with $x(0) \in V$ is defined and in $U$ for all $t \geq 0$. If $V$ can be chosen so that additionally $\lim_{t\to\infty} x(t) = e$ whenever $x(0) \in V$, then $e$ is **asymptotically stable** [HS74, pp. 185–186]. See Figure 1. The equilibrium $e$ is **unstable** if it is not stable.

How can one determine whether $e$ is asymptotically stable? If $f$ is $C^1$, the simplest test is Lyapunov's indirect method [Lya92], whose statement below includes complex eigenvalues of the derivative $d_e f$ of $f$ at $e$ (see [HS74, pp. 181, 187]).[2]

**Proposition 1** (Lyapunov)**.** *If $f$ is $C^1$ and all eigenvalues of $d_e f$ have negative real parts, then $e$ is asymptotically stable. If at least one eigenvalue of $d_e f$ has positive real part, then $e$ is unstable.*

Proposition 1 is quite useful, but it provides no information when all of the eigenvalues have nonpositive real part and at least one has zero real part. This limitation is addressed by Lyapunov's direct method [Lya92]. In its statement below, $\dot{L}$ denotes the function $\dot{L}(x) = d_x L(f(x))$ defined where $L$ is differentiable (see [HS74, p. 193]).

**Proposition 2** (Lyapunov)**.** *If there is an open neighborhood $U$ of $e$ and a continuous function $L\colon U \to [0, \infty)$ such that*

- $L^{-1}(0) = \{e\}$,
- $L$ *is differentiable on* $U \setminus \{e\}$, *and*
- $\dot{L} \leq 0$ *on* $U \setminus \{e\}$,

*then $e$ is stable. If additionally $\dot{L} < 0$ on $U \setminus \{e\}$, then $e$ is asymptotically stable.*

A function satisfying the properties in the first sentence of the proposition is called a **Lyapunov function** for $e$, and a **strict Lyapunov function** if it also

[2] Strictly speaking, these are eigenvalues of $T_eM \xrightarrow{d_e f} T_{0_e}TM = T_eM \oplus T_eM \xrightarrow{\mathrm{Pr}_2} T_eM$.

satisfies the property in the second sentence. Note that there are situations in which asymptotic stability may be deduced using non-strict Lyapunov functions [LaS60].

A natural question arises: if stability is present, does there always exist a Lyapunov function certifying it according to Proposition 2? For stability without asymptotic stability, simple examples show that the answer is "no".

**Example 1.** It is readily seen that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} \sin^2(\pi/x), & x > 0, \\ 0, & x \leq 0 \end{cases}$$

is smooth and, since $f^{-1}(0) = (-\infty, 0] \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$, 0 is a stable but not asymptotically stable equilibrium of $\dot{x} = f(x)$. However, a Lyapunov function $L$ for 0 cannot exist: since $f > 0$ on $(\frac{1}{n+1}, \frac{1}{n})$ for $n \in \mathbb{N}$, such an $L$ would satisfy $L(\frac{1}{n+m}) \geq L(\frac{1}{n})$ for all sufficiently large $m, n \in \mathbb{N}$ and hence $0 = L(0) \geq L(\frac{1}{n}) > 0$, a contradiction.

The situation is different for asymptotic stability. Assume that $e$ is asymptotically stable and let $B \subset M$ denote its **basin of attraction**: the largest set for which every solution $x(t)$ to (1) with $x(0) \in B$ is defined for all $t \geq 0$ and satisfies $\lim_{t\to\infty} x(t) = e$. Every solution starting in $B$ remains in $B$, and $B$ is an open set since solutions depend continuously on their initial conditions [Har82, Theorem V.II.1]. A **proper strict Lyapunov function** is a strict Lyapunov function $L\colon B \to [0, \infty)$ with compact sublevel sets $L^{-1}([0, c])$ for all $c \geq 0$. The following partial converse to Proposition 2 was proved separately by Kurzweil and Massera [Kur56, Mas56].

**Theorem 1** (Kurzweil, Massera)**.** If $e$ is asymptotically stable, then a *smooth* proper strict Lyapunov function exists.

Thus, when asymptotic stability is present, one can always certify it according to Proposition 2: a proper strict Lyapunov function always exists. Moreover, such a function may be chosen smooth ($C^\infty$). This does not mean that explicitly finding such a function is easy. However, it does provide a bridge between asymptotic stability theory and differential topology that will be exploited in later sections.

There are analogous instability tests based on finding suitable auxiliary functions ([Che61, p. 27], [Kra63, Theorem 7.1]). In keeping with the theme of this paper, we instead conclude this section with topological necessary conditions for asymptotic stability whose statements do not involve any auxiliary functions.

First, consider any isolated zero $z \in M$ of a continuous vector field $g$ on $M$. Let $\varphi\colon U \to V$ be a diffeomorphism from an open neighborhood $U \subset M$ of $z$ to an open set $V \subset \mathbb{R}^n$, where $U$ contains no other zeros of $g$. Then $d\varphi \circ g \circ \varphi^{-1}$ is a continuous vector field on $V$ corresponding to a map $h\colon V \to \mathbb{R}^n$. The **index** $\operatorname{ind}_z(g)$ of $g$ at $z$ is defined to be the degree [Hir94, p. 124] of the continuous map

$$h/|h|\colon \partial D \to S^{n-1}$$

for any $n$-disc $D \subset V$ centered at $\varphi(z)$, where $S^{n-1} = \{x \in \mathbb{R}^n \colon |x| = 1\}$ is the unit sphere. It is independent of the choices made ([Mil97, pp. 32–35], [Hir94, p. 124]).

Next, recall that the **Euler characteristic** $\chi(N)$ of a compact smooth manifold with boundary $N$ may be defined as the usual finite alternating sum

$$\chi(N) = \#\,\text{vertices} \; - \; \#\,\text{edges} \; + \; \#\,\text{faces} \; - \; \ldots,$$

with respect to any finite triangulation [Mun66, Theorem II.10.6]. We use the following statement of the Poincaré–Hopf theorem [Poi85, Hop27] (see [Mil97, Pug68]).

**Lemma 1** (Poincaré, Hopf)**.** Let $N$ be a compact smooth manifold with boundary. Let $g$ be a continuous vector field on $N$ that points outward on the boundary $\partial N$ and has only finitely many zeros $z_1, \ldots, z_m$, all contained in the interior of $N$. Then

$$\chi(N) = \sum_{i=1}^{m} \operatorname{ind}_{z_i}(g).$$

The first necessary condition below goes back to Bobylev and Krasnosel'skiĭ [BK74] (see [KZ84, Theorem 52.1]), and the second is a special case of an extension [KK22, Theorem 3.2] of one due to Brockett [Bro83, Theorem 1.(iii)] (see Zabczyk for the continuous case [Zab89, p. 1, Theorem 3]).

**Proposition 3.** Assume that $e$ is asymptotically stable. Then $\operatorname{ind}_e(-f) = 1$. Additionally, for any neighborhood $U \subset M$ of $e$ and any sufficiently small continuous vector field $g$ on $U$, there exists $y \in U$ such that $f(y) = g(y)$.

**Remark 1.** In contrast to the first statement, on manifolds of dimension larger than 2, the index of a smooth vector field at an isolated (non-asymptotically) stable equilibrium can take any integer value [BV02, Theorem A].

*Proof.* By Theorem 1, we may choose a smooth proper strict Lyapunov function $L\colon B \to [0,\infty)$. Since $L$ is proper, there is $c > 0$ such that $N := L^{-1}([0,c]) \subset U$. Since $dL$ is nonzero on $B \setminus \{e\}$, the implicit function theorem implies that $N$ is a compact smooth manifold with boundary, and $\partial N = L^{-1}(c)$.

Letting $t \mapsto \Phi(t, x_0)$ be the maximal solution to (1) satisfying $x(0) = x_0 \in B$, we obtain a continuous map $(t,x) \mapsto \Phi(t,x)$ defined on an open neighborhood of $[0,\infty) \times B$ in $\mathbb{R} \times B$ [Har82, Theorem V.2.1]. We have a homotopy $I \times N \to N$,

$$(t,x) \mapsto \begin{cases} \Phi(\frac{t}{1-t}, x), & 0 \le t < 1, \\ e, & t = 1 \end{cases} \tag{2}$$

from $\operatorname{id}_N$ to a constant map, so it follows that $\chi(N) = 1$ [Hat02, Chapter 2].

Since $-f$ points outward on $\partial N$ and the only zero of $-f$ in $N$ is $e$, it follows from Lemma 1 that $\operatorname{ind}_e(-f) = \chi(N) = 1$.

Since $g - f$ points outward on $\partial N$ for any sufficiently small continuous vector field $g$ on $U$, Lemma 1 also implies that $g - f$ has a zero $y$ in the interior of $N$ for any such $g$. In particular, $y \in U$ and $g(y) = f(y)$. □

Using the second statement of Proposition 3, it is often easy to prove that an equilibrium $e$ is *not* asymptotically stable by producing arbitrarily small vector fields $g$ satisfying $f(x) \neq g(x)$ for all $x$ near $e$. When $M = \mathbb{R}^n$ and $f$, $g$ are viewed as $\mathbb{R}^n$-valued, considering constant vector fields $g$ yields that $e$ is not asymptotically stable if $f(U) \subset \mathbb{R}^n$ is not a neighborhood of 0 for any neighborhood $U$ of $e$. The latter criterion was introduced by Brockett for the more general problem of feedback stabilizability of control systems [Bro83, Theorem 1.(iii)], and the former was introduced for more general problems in [KK22, Theorem 3.2; see also Section 6.1].

When $M = \mathbb{R}^n$, $f$ is viewed as $\mathbb{R}^n$-valued, and $e \in \mathbb{R}^n$ is an isolated equilibrium, the first statement of Proposition 3 is equivalent to the homomorphism

$$\mathbb{Z} \cong H_{n-1}(D \setminus \{e\}) \to H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$$

on singular homology induced by $-f$ being the identity map for any sufficiently small $n$-disc $D$ centered at $e$. Coron made this observation and used it to introduce a necessary condition for feedback stabilizability of control systems stronger than Brockett's [Cor90, Equation (15), Theorem 2, p. 229] (at least in the present setting [Kva25b]).

See [Son99, JM23] for more on asymptotic stability and feedback stabilizability tests, and [Wil69, Kel15, FP19, Fat22] for more on converse Lyapunov theorems.

## 3. Basins of attraction

In this section, following Wilson [Wil67, Section 2], we show that the basin of attraction of an asymptotically stable equilibrium is diffeomorphic to Euclidean space. To illustrate techniques of differential topology, we give Milnor's proof [Mil64, Lemma 3] of a requisite theorem of Brown and Stallings [Bro61, Sta62]. Here and later, we rely on the disc theorem of Palais and Cerf [Pal60, Cer61].

We recall some definitions. Let $M$, $N$ be smooth manifolds with boundary, such as the standard $n$-disc $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ or unit interval $I = [0, 1]$. A map $F : N \to M$ is a (smooth) **embedding** if $F(N) \subset M$ is a smooth submanifold with boundary [Hir94, p. 30] and $F : N \to F(N)$ is a diffeomorphism. A smooth homotopy $H : I \times N \to M$ is an **isotopy** if $H_t = H(t, \cdot) : N \to M$ is an embedding for each $t \in I$, and an **ambient isotopy** if $M = N$, $H_0 = \mathrm{id}_M$, and $H_t$ is a diffeomorphism for each $t \in I$ [Hir94, p. 178].

The following statement of the disc theorem includes a refinement useful for later purposes [Kos93, Theorem III.3.6].

**Lemma 2** (Palais, Cerf)**.** Let $F$ and $G$ be embeddings of $D^k$ into the interior of a connected smooth $n$-manifold with boundary $M$. If $k = n$ and $M$ is orientable, assume that $F$ and $G$ both preserve, or both reverse, orientation. Then there is an ambient isotopy $H : I \times M \to M$ satisfying $H_1 \circ F = G$. Moreover, if $F(0) = G(0)$, then $H$ may be chosen so that $H_t \circ F(0) = F(0)$ for all $t \in I$.

We now prove a smooth version of a theorem of Brown and Stallings [Bro61, Sta62] following Milnor [Mil64, Lemma 3].

**Lemma 3** (Brown, Stallings)**.** Let $M$ be a smooth $n$-manifold such that each compact subset of $M$ is contained in an open subset diffeomorphic to $\mathbb{R}^n$. Then $M$ itself is diffeomorphic to Euclidean space.

*Proof.* Let $(K_i)_{i=1}^{\infty}$ be an exhaustion of $M$ by compact subsets $K_i \subset M$, so that $M = \bigcup_i K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ for each $i$ [Lee13, Proposition A.60]. Let $W_1 \subset M$ be any smooth submanifold with boundary diffeomorphic to $D^n$. Since $W_1$ is compact, there exists $\ell \geq 1$ such that $W_1 \subset \operatorname{Int} K_\ell$. Our assumption implies that there is a smooth submanifold with boundary $W_2$ diffeomorphic to $D^n$ such that $K_\ell \subset \operatorname{Int} W_2$ and hence also $K_1, W_1 \subset \operatorname{Int} W_2$. Continuing inductively, we obtain a sequence

$$W_1 \subset W_2 \subset W_3 \subset \cdots \subset M$$

of submanifolds with boundary, each diffeomorphic to $D^n$, so that $K_i, W_i \subset \operatorname{Int} W_{i+1}$ for all $i \in \mathbb{N}$. In particular, $M = \bigcup_i W_i$.

We will compare this with the sequence

$$D_1^n \subset D_2^n \subset D_3^n \subset \cdots \subset \mathbb{R}^n$$

where $D_i^n \subset \mathbb{R}^n$ is the $n$-disc of radius $i$ centered at 0.

Fix a diffeomorphism $F_1 \colon D_1^n \to W_1$. Choose a diffeomorphism $G \colon D_2^n \to W_2$ so that $G|_{D_1^n}$ and $F_1$ have the same orientation as embeddings into $W_2$. Lemma 2 provides an ambient isotopy $H \colon I \times W_2 \to W_2$ satisfying $H_1 \circ G|_{D_1^n} = F_1$, so the diffeomorphism $F_2 \colon D_2^n \to W_2$ defined by $F_2 = H_1 \circ G$ satisfies $F_2|_{D_1^n} = F_1$. Continuing inductively, we obtain diffeomorphisms $F_i \colon D_i^n \to W_i$ satisfying $F_{i+1}|_{D_i^n} = F_i$ for all $i \in \mathbb{N}$. Piecing these together, we obtain the desired diffeomorphism $\mathbb{R}^n \to M$. □

We now prove Wilson's result that basins of attraction are diffeomorphic to Euclidean space [Wil67, Theorem 3.4].

**Theorem 2** (Wilson)**.** Let $f$ be a uniquely integrable continuous vector field on a smooth $n$-manifold $M$, and $e \in M$ be an asymptotically stable equilibrium for (1). Then the basin of attraction $B \subset M$ is diffeomorphic to $\mathbb{R}^n$.

*Proof.* By Theorem 1, there is a smooth proper strict Lyapunov function $L \colon B \to [0, \infty)$. Let $\nabla L$ be the gradient of $L$ with respect to a smooth Riemannian metric on $M$, and $\rho \colon B \to (0, \infty)$ be a smooth function such that the vector field $-\rho \nabla L$ is complete and generates a smooth flow $\Phi \colon \mathbb{R} \times B \to B$ [Har83, Corollary 2]. Then for any $x_0 \in B$, $t \mapsto \Phi^t(x_0) = \Phi(t, x_0)$ is the maximal solution to the initial value problem

$$\dot{x} = -\rho(x) \nabla L(x), \quad x(0) = x_0.$$

Moreover, $e$ is also asymptotically stable for $-\rho \nabla L$ with basin of attraction $B$.

Fix an open neighborhood $U \subset B$ of $e$ that is diffeomorphic to $\mathbb{R}^n$. If $K \subset B$ is an arbitrary compact set, asymptotic stability implies that there is $T > 0$ such that $\Phi^t(K) \subset U$ for all $t \geq T$. Hence $\Phi^{-T}(U)$ is an open neighborhood of $K$ that is diffeomorphic to $\mathbb{R}^n$. Thus, Lemma 3 implies that $B$ is diffeomorphic to $\mathbb{R}^n$. □

**Remark 2.** In the control literature, a well-known result of Bhat and Bernstein asserts that $B$ cannot be diffeomorphic to a vector bundle over a compact manifold of positive dimension [BB00, Theorem 1]. This also follows from Wilson's Theorem 2, since $\mathbb{R}^n$ is contractible (defined below) while any such vector bundle is not.

## 4. Level and sublevel sets of Lyapunov functions

In this section, we study the nonzero level and sublevel sets of smooth proper strict Lyapunov functions (as in Theorem 1). If the ambient manifold has dimension $n$, then, as we will see, the level sets are homeomorphic to $S^{n-1}$, and diffeomorphic to $S^{n-1}$ if $n \neq 5$. Moreover, the sublevel sets are homeomorphic to $D^n$, and diffeomorphic to $D^n$ if $n \neq 4, 5$. Applications are given in this section and the next.

We rely on Smale's proof of the topological Poincaré conjecture for smooth manifolds in dimensions greater than 4 and his $h$-cobordism theorem [Sma61, Sma62], Freedman's proof of the topological Poincaré conjecture in dimension 4 [Fre82], the generalized Schoenflies theorem of Mazur and Brown [Maz59, Bro60], and Perelman's resolution of the Poincaré conjecture in dimension 3 [Per02, Per03b, Per03a]. We summarize the consequences we need in the following lemma.

Recall that a path-connected space $X$ is **simply connected** if any continuous map $S^1 \to X$ is homotopic to a constant map, and **contractible** if the identity map $\mathrm{id}_X$ is homotopic to a constant map. A pair of spaces $X$ and $Y$ are **homotopy equivalent** if there are continuous maps $F \colon X \to Y$ and $G \colon Y \to X$ such that $G \circ F$ and $F \circ G$ are homotopic to the identity maps of $X$ and $Y$.

**Lemma 4.** Let $N$ be a compact smooth $n$-manifold with boundary. Assume that $N$ is contractible and its boundary $\partial N$ is homotopy equivalent to $S^{n-1}$. Then $N$ is homeomorphic to $D^n$, and diffeomorphic to $D^n$ if $n \neq 4, 5$. Moreover, $\partial N$ is homeomorphic to $S^{n-1}$, and diffeomorphic to $S^{n-1}$ if $n \neq 5$.

*Proof.* It is a standard consequence of Smale's $h$-cobordism theorem that $N$ is diffeomorphic to $D^n$ for $n > 5$ [Sma62, Theorem 5.1], and the same holds for $n < 3$ by the classification of manifolds of dimensions 1 and 2 ([Mil97, Appendix], [Hir94, Theorem 9.3.11]). This proves the theorem for $n \neq 3, 4, 5$.

Since $\partial N$ is homotopy equivalent to $S^{n-1}$, the classification of manifolds of dimension 2 implies that $\partial N$ is diffeomorphic to $S^{n-1}$ for $n = 3$ [Hir94, Theorem 9.3.11], and the same holds for $n = 4$ by Perelman's resolution of the Poincaré conjecture in dimension 3 [Per02, Per03b, Per03a] (see [MT07, Corollary 0.2]). For $n = 5$, $\partial N$ is at least homeomorphic to $S^{n-1}$ by Freedman's resolution of the topological Poincaré conjecture in dimension 4 [Fre82, Theorem 1.6].

For $n = 5$, it is then a standard consequence of Smale's proof of the 5-dimensional Poincaré conjecture [Sma61, Theorem A] and the generalized Schoenflies theorem of Mazur and Brown [Bro62, Theorem 4] that $N$ is homeomorphic to $D^n$ (see [Mil65, Proposition C]). This proves the theorem for $n = 5$.

For $n = 3, 4$, following [Mil65, p. 111], we construct a compact orientable smooth $n$-manifold $M$ by gluing $D^n$ to $N$ along a diffeomorphism $\partial D^n \to \partial N$. By the Seifert–van Kampen and Mayer–Vietoris theorems, $M$ is simply connected with the homology of $S^n$ [Hat02, Theorem 1.20, p. 149], so the Hurewicz theorem yields a continuous map $F\colon S^n \to M$ inducing isomorphisms on homology [Hat02, Theorem 4.37]. Since $M$ is homotopy equivalent to a CW complex [Hat02, Theorem A.12], it follows from the homology version of Whitehead's theorem that $F$ is a homotopy equivalence [Hat02, Corollary 4.33].

Thus, there is a map $G\colon M \to S^n$ that is a diffeomorphism for $n = 3$ by Perelman's work, and a homeomorphism for $n = 4$ by Freedman's. For $n = 3$, Lemma 2 provides a diffeomorphism $H\colon S^n \to S^n$ such that $H \circ G(D^n)$ is the upper hemisphere and hence $H \circ G(N)$ is the lower hemisphere, so $N$ is diffeomorphic to $D^n$ if $n = 3$. And for $n = 4$, the generalized Schoenflies theorem provides a homeomorphism $H\colon S^n \to S^n$ with the same properties [Bro62, Theorem 4], so $N$ is homeomorphic to $D^n$ for $n = 4$. This completes the proof. ∎

Now let $M$ be a smooth $n$-manifold, $f$ be a uniquely integrable continuous vector field on $M$, $e$ be an asymptotically stable equilibrium for the system (1) with basin of attraction $B$, and $L\colon B \to [0, \infty)$ be any smooth proper strict Lyapunov function.

**Theorem 3.** Fix any $b, c > 0$. The sublevel set $L^{-1}([0, b])$ and level set $L^{-1}(b)$ are respectively diffeomorphic to $L^{-1}([0, c])$ and $L^{-1}(c)$. Moreover, $L^{-1}([0, c])$ is homeomorphic to $D^n$, and diffeomorphic to $D^n$ if $n \neq 4, 5$. Furthermore, $L^{-1}(c)$ is homeomorphic to $S^{n-1}$, and diffeomorphic to $S^{n-1}$ if $n \neq 5$.

**Remark 3.** Wilson proved that the level sets are homotopy equivalent to $S^{n-1}$, and diffeomorphic to $S^{n-1}$ if $n \neq 4, 5$ [Wil67, p. 325, Theorem 3.4]. Grüne, Wirth, and Sontag noted that the sublevel sets are diffeomorphic to $D^n$ if $n > 5$, and the level sets are homeomorphic to $S^{n-1}$ if $n = 5$ [GSW99, p. 130]. Byrnes observed that the sublevel sets are homeomorphic to $D^n$ [Byr08, Theorem 3.1], and Jongeneel noted that the level sets are diffeomorphic to $S^{n-1}$ if $n = 3$ [Jon26, p. 5]. The heart of the proof is contained in the references cited to prove Lemma 4.

*Proof.* As in the proof of Theorem 2, let $\nabla L$ be the gradient of $L$ with respect to a smooth Riemannian metric on $M$, $\rho\colon B \to (0,\infty)$ be a smooth function such that $-\rho\nabla L$ is complete, and $\Phi\colon \mathbb{R}\times B \to B$ be the smooth flow of $-\rho\nabla L$. Then $e$ is also asymptotically stable for $-\rho\nabla L$ with basin of attraction $B$.

Consider any $a > 0$. Since $\nabla L$ is nonzero on $B\setminus\{e\}$, the implicit function theorem implies that the sublevel set $N_a = L^{-1}([0,a])$ is a compact smooth $n$-manifold with boundary, and $\partial N_a = L^{-1}(a)$. Each $\Phi$-trajectory crosses $\partial N_a$ exactly once and transversely, so the map

$$\mathbb{R}\times \partial N_a \to B\setminus\{e\}, \quad (t,x)\mapsto \Phi(-t,x) \tag{3}$$

is a diffeomorphism. Let $(\tau_a,\rho_a)\colon B\setminus\{e\}\to \mathbb{R}\times\partial N_a$ denote its inverse.

Define a smooth map $F\colon I\times \partial N_c \to B\setminus\{e\}$ by $F(t,x) = F_t(x) = \Phi(t\tau_b(x),x)$. Since $\rho_c\circ F_t = \mathrm{id}_{\partial N_c}$, the map $F_t$ is an embedding for all $t\in I$. In particular, $\partial N_c$ is diffeomorphic to $\partial N_b = F_1(\partial N_c)$. The isotopy extension theorem provides an ambient isotopy $G\colon I\times B\to B$ satisfying $G_t\circ F_0 = F_t$ and $G_t(e) = e$ for all $t\in I$, where $G_t = G(t,\cdot)$ [Hir94, Theorem 8.1.3]. In particular, $\partial N_b = G_1(\partial N_c)$, so the diffeomorphism $G_1$ also maps the connected component $\operatorname{Int} N_c$ of $B\setminus\partial N_c$ containing $e$ to the connected component $\operatorname{Int} N_b$ of $B\setminus\partial N_b$ containing $e$. Thus, $G_1$ restricts to a diffeomorphism from $N_c$ to $N_b$.

Next, existence of the diffeomorphism (3) implies that $\partial N_c$ is homotopy equivalent to $B\setminus\{e\}$, which is diffeomorphic to $\mathbb{R}^n\setminus\{0\}$ by Theorem 2. Since $\mathbb{R}^n\setminus\{0\}$ is diffeomorphic to $\mathbb{R}\times S^{n-1}$, it follows that $\partial N_c$ is homotopy equivalent to $S^{n-1}$. Finally, we showed in the proof of Proposition 3 that $N_c$ is contractible (via the homotopy (2)), so applying Lemma 4 completes the proof. □

We now give two applications of Theorem 3. The first extends the Hartman–Grobman theorem [Har60, Gro59] to nonhyperbolic but asymptotically stable equilibria of vector fields, and the second is related to a question of Conley [Con78].

Recall that $f$ is **complete** if all maximal solutions $x(t)$ of (1) are defined on $\mathbb{R}$.

**Proposition 4.** If $f$ is complete, there is a homeomorphism $H\colon B\to\mathbb{R}^n$ satisfying

$$H(x(t)) = e^{-t}H(x(0)) \tag{4}$$

for all solutions $x(t)$ of (1). Moreover, if $n\neq 5$ and $f$ is $C^k$ with $k\in\mathbb{N}\cup\{\infty\}$, then $H$ may be chosen to restrict to a $C^k$ diffeomorphism $B\setminus\{e\}\to\mathbb{R}^n\setminus\{0\}$.

**Remark 4.** The first trace of this result seems to be in work of Coleman, who proved a local version of the first statement under the assumption that there is a Lyapunov function with level sets homeomorphic to spheres [Col65, Col66] (see [Wil78, p. 247]). The proof below follows [KS25, Theorem 2], which in turn follows one sketched by Grüne, Wirth, and Sontag for $n\neq 4$ [GSW99, p. 133].

**Remark 5.** For the second statement, the proof requires only that $f$ generates a $C^k$ flow on the complement of the equilibrium. (The vector field generating a $C^k$ flow need only be $C^{k-1}$ [Har83].)

**Remark 6.** Techniques similar to those in the proof below can be used to prove an extension of the Morse lemma [Mor96] (see [Mil63, Lemma 2.2]) to local minima of non-Morse functions, as contained in a result of Grüne, Wirth, and Sontag [GSW99, Proposition 1] (see also [KS25, p. 3]).

*Proof.* Define $N = L^{-1}([0,1])$, so that $\partial N = L^{-1}(1)$. By Theorem 3, we may fix a homeomorphism $F\colon \partial N \to S^{n-1}$, and we may require it to be a ($C^\infty$) diffeomorphism if $n \neq 5$. Let $\Phi\colon \mathbb{R} \times B \to B$ be the flow generated by $f|_B$, so that $t \mapsto \Phi(t, x_0)$ is the maximal solution to (1) satisfying $x(0) = x_0$.

Similarly to the proof of Theorem 3, the map

$$\mathbb{R} \times \partial N \to B \setminus \{e\}, \quad (t,x) \mapsto \Phi(-t, x) \tag{5}$$

is a homeomorphism, and a $C^k$ diffeomorphism if $\Phi$ is $C^k$. Let the inverse be denoted by $(\tau, \rho)\colon B \setminus \{e\} \to \mathbb{R} \times \partial N$. Since $\tau(x) \to -\infty$ as $x \to e$, the map $H\colon B \to \mathbb{R}^n$ defined by $H(e) = 0$ and

$$H(x) = e^{\tau(x)} F(\rho(x))$$

for $x \in B \setminus \{e\}$ is a homeomorphism that, if $n \neq 5$ and $\Phi$ is $C^k$, restricts to a $C^k$ diffeomorphism $B \setminus \{e\} \to \mathbb{R}^n \setminus \{0\}$. And (4) holds since $\rho \circ \Phi^t|_{B\setminus\{e\}} = \rho$ and $\tau \circ \Phi^t|_{B\setminus\{e\}} = \tau - t$ for all $t \in \mathbb{R}$. □

Given a pair of dynamical systems and invariant subsets thereof, it is natural to wonder whether one pair can be deformed, or *continued*, into the other. In a topological setting, Conley proved that the isolated invariant sets must have isomorphic Conley indices for such a continuation to exist [Con78, Section IV.2], and he asked to what extent the converse is true [Con78, Section IV.8.1.A]. We refer the reader to [Con78, pp. 51, 65] for the definitions; here we simply note that asymptotically stable equilibria have isomorphic Conley indices, and the following result gives a continuation with particularly strong properties. It gives a partial affirmative answer in the special case of globally asymptotically stable equilibria, which, to the author's knowledge, does not appear in the literature (but see Remark 9).

For the statement, recall that a **flow** is a map $\Phi\colon \mathbb{R} \times M \to M$ satisfying $\Phi^0 = \mathrm{id}_M$ and $\Phi^{s+t} = \Phi^s \circ \Phi^t$ for all $s, t \in \mathbb{R}$, where $\Phi^t = \Phi(t, \cdot)\colon M \to M$. An equilibrium is a fixed point of $\Phi^t$ for all $t \in \mathbb{R}$. Asymptotic stability and basins of attraction of such equilibria are defined exactly as in Section 2, using $\Phi$-trajectories $t \mapsto \Phi^t(x_0)$ in place of solutions to (1). An equilibrium is **globally asymptotically stable** if its basin of attraction is all of $M$. A path $\{\Phi_s\}_{s\in I}$ of flows $\Phi_s\colon \mathbb{R} \times M \to M$ is **continuous** if the associated map $(s,t,x) \mapsto \Phi_s^t(x)$ is.

**Proposition 5.** Let $g$, $h$ be continuous and uniquely integrable vector fields with globally asymptotically stable equilibria $e_g, e_h \in M$. Assume that $g$, $h$ are complete, so that they generate continuous flows $\Phi_g, \Phi_h\colon \mathbb{R} \times M \to M$. Then there exists a continuous path $\{\Phi_s\}_{s\in I}$ of flows from $\Phi_0 = \Phi_g$ to $\Phi_1 = \Phi_h$ such that $\Phi_s$ has a globally asymptotically stable equilibrium $e_s$ for all $s \in I$. Moreover, if $e_g = e_h$, then we may arrange that $e_s = e_g$ for all $s \in I$.

**Remark 7.** Every continuous vector field that is uniquely integrable and complete generates a continuous flow [Har82, Theorem V.II.1], but the converse does not hold, leading to challenges for smooth converse Lyapunov theorems [FP19, Section 7].

**Remark 8.** The path $\{\Phi_s\}_{s\in I}$ constructed is of the form $\Phi_s^t = H_s \circ \Phi_g^t \circ H_s^{-1}$, where $H_s = H(s, \cdot)$ for a **topological ambient isotopy** $H\colon I \times M \to M$, meaning $H$ is continuous, $H_0 = \mathrm{id}_M$, and $H_s\colon M \to M$ is a homeomorphism for each $s \in I$.

**Remark 9.** If $g$, $h$ are not complete, then they still generate semiflows (by the definition of global asymptotic stability), and the conclusion of Proposition 5 still holds with "flow" replaced by "semiflow". To prove this, first homotopically rescale

$g$, $h$ to be complete (cf. [Har83, Corollary 2]) and then apply the proposition. This extends a theorem of Jongeneel, which assumed that $n \neq 5$ and $g$, $h$ are locally Lipschitz [Jon26, Theorem 3.6], resolving [Jon26, open problem 2].

*Proof.* By Theorem 2, we may assume that $M = \mathbb{R}^n$. Then by Proposition 4, there are homeomorphisms $F_g, F_h \colon \mathbb{R}^n \to \mathbb{R}^n$ such that, for all $t \in \mathbb{R}$,

$$F_g \circ \Phi_g^t = e^{-t} \circ F_g \quad \text{and} \quad F_h \circ \Phi_h^t = e^{-t} \circ F_h.$$

Since any linear map commutes with $e^{-t}$, we may assume that $F_g$, $F_h$ are orientation-preserving after post-composing each with a reflection of $\mathbb{R}^n$ if necessary.

The homeomorphism $G \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $G = F_h^{-1} \circ F_g$ satisfies $\Phi_h^t = G \circ \Phi_g^t \circ G^{-1}$ for all $t \in \mathbb{R}$. Since $G$ is orientation-preserving, the stable homeomorphism theorem provides a topological ambient isotopy $H \colon I \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying $H_1 = G$ [Kir69, Qui82] (see [Kir69, p. 576], [Edw84, p. 247]). The formula

$$\Phi_s^t = H_s \circ \Phi_g^t \circ H_s^{-1}$$

then defines a continuous path $\{\Phi_s\}_{s \in I}$ of flows from $\Phi_0 = \Phi_g$ to $\Phi_1 = \Phi_h$ such that $\Phi_s$ has the globally asymptotically stable equilibrium $e_s = H_s(e_g)$ for all $s \in I$. If $e_g = e_h$, then $H_1(e_g) = e_g$, and we instead replace $H$ with the topological ambient isotopy $\tilde{H} \colon I \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\tilde{H}_s = H_s - e_s + e_g.$$

□

## 5. The space of asymptotically stable vector fields

In Sections 3 and 4, we studied spaces related to asymptotically stable vector fields. In this section, we instead study the function space of all such vector fields.

More precisely, we say that a continuous and uniquely integrable vector field is **asymptotically stable** if it has a globally asymptotically stable equilibrium. As we know from Theorem 2, the domain of such a vector field is diffeomorphic to Euclidean space, so we need only consider asymptotically stable vector fields on $\mathbb{R}^n$. Fixing $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we define

$$\mathcal{S}^k(\mathbb{R}^n) = \{C^k \text{ asymptotically stable vector fields on } \mathbb{R}^n\}$$

and equip this space with the $C^k$ topology, also known as the compact-open or weak $C^k$ topology. This is the topology of uniform convergence on compact sets of not only the functions but also all derivatives up to order $k$ [Hir94, Section 2.1]. As evidenced by the applications to be discussed, this is the natural choice.

Recall that a **weak homotopy equivalence** is a continuous map between spaces that induces a bijection on path-components and isomorphisms on all homotopy groups [Hat02, p. 340, 352]. Weak homotopy equivalences also induce isomorphisms on all homology and cohomology groups [Hat02, Proposition 4.21]. Spaces $X$ and $Y$ have the same **weak homotopy type** if they are equivalent in the equivalence relation generated by weak homotopy equivalences, so are connected by a zigzag

$$X \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \rightarrow Z_{m-1} \leftarrow Z_m \rightarrow Y$$

in which each map is a weak homotopy equivalence.

Our goal is to motivate and sketch the proofs of the following pair of results.

**Theorem 4** ([Kva25a])**.** If $n \neq 4, 5$, then $\mathcal{S}^k(\mathbb{R}^n)$ has the same weak homotopy type as the nonlinear Grassmannian

$$\mathrm{Gr}(D^n, \mathbb{R}^n) = \mathrm{Emb}(D^n, \mathbb{R}^n)/\mathrm{Diff}(D^n)$$

of submanifolds of $\mathbb{R}^n$ diffeomorphic to the $n$-disc $D^n$.

Here $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is topologized as the quotient of the space $\mathrm{Emb}(D^n, \mathbb{R}^n)$ of $C^\infty$ embeddings $D^n \hookrightarrow \mathbb{R}^n$ by the natural right action of the group $\mathrm{Diff}(D^n)$ of diffeomorphisms of $D^n$, where the latter spaces have the $C^\infty$ topology [GBV14].

In particular, $\mathcal{S}^k(\mathbb{R}^n)$ and $\mathrm{Gr}(D^n, \mathbb{R}^n)$ have isomorphic homotopy, homology, and cohomology groups if $n \neq 4, 5$. Recall that a path-connected space $X$ is simply connected if and only if its first homotopy group is trivial, and **weakly contractible** if all of its homotopy groups are trivial.

**Corollary 1.** The space $\mathcal{S}^k(\mathbb{R}^n)$ is both path-connected and simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$.

The corollary follows from the theorem and the fact that $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is path-connected for all $n$, simply connected if $n \neq 4, 5$, and weakly contractible if $n < 4$. Path-connectedness follows from the disc theorem (Lemma 2). Weak contractibility for $n < 4$ follows from Smale's result that the space $\mathrm{Diff}_\partial(D^2)$ of diffeomorphisms of the 2-disc restricting to the identity on $\partial D^2$ is contractible [Sma59], and Hatcher's proof of the analogous Smale conjecture for $D^3$ [Hat83]. Simple connectivity for $n > 5$ follows from Cerf's proof of the pseudo-isotopy theorem, in which he developed one-parameter versions of Smale's handle-theoretic methods [Cer70]. These implications are explained in detail at the end of this section (see Lemma 5) using techniques from the study of embedding spaces [ABK72, Bud08, CMRW17, Kup19].

**Remark 10.** By a theorem of Hirsch, any smooth 4-manifold homeomorphic to $S^4$ is a nonzero level set of some smooth proper strict Lyapunov function [Hir65, Theorem 2]. Using this result, it can be shown that $\mathcal{S}^k(\mathbb{R}^5)$ is path-connected if and only if the 4-dimensional smooth Poincaré conjecture is true [Kva25a, Proposition 2], which is also true if and only if the second statement of Proposition 4 holds for $n = 5$ [KS25, Proposition 1].

**Remark 11.** Theorem 4 and Corollary 1, as well as later results in this section, also hold for the subspaces $\mathcal{S}^k_e(\mathbb{R}^n) \subset \mathcal{S}^k(\mathbb{R}^n)$ of asymptotically stable vector fields with specified equilibrium $e \in \mathbb{R}^n$. The results also hold for the subspaces of (backward) complete vector fields. (Forward completeness follows from asymptotic stability.)

Before sketching the proof of Theorem 4, we motivate it with some applications.

### 5.1. Applications.

5.1.1. *Boundary value problems.* Asymptotic stability models a kind of robust steady-state behavior that is useful for many real-world systems. In particular, a central goal in control theory is to achieve this property by means of a feedback control law [Son98, Cor07, Zab20, JM23].

Just as one wants to tune an elevator so that it comes smoothly to rest, it is natural to design stabilizing feedback laws depending on parameters that can be adjusted to produce desired transient behaviors. This leads to the following.

**Question.** If stabilizing feedback laws have been designed for *some* parameter values, when can the design be extended to *all* parameter values?

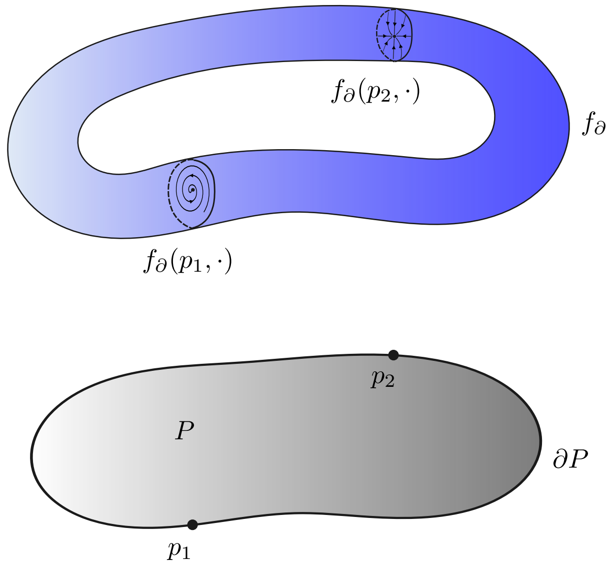


FIGURE 2. Proposition 6 gives conditions ensuring that any $C^k$ family of asymptotically stable vector fields parametrized by $p \in \partial P$ admits an extension to such a $C^k$ family parametrized by $p \in P$.

Smooth extensions of such designs are of particular interest. In certain cases, the following result guarantees existence of solutions to this smooth extension problem or, equivalently, boundary value problem. See Figure 2. For the statement, a family $\{f_p\}_{p\in P}$ of maps $f_p\colon X \to Y$ is $C^k$ if the map $(p,x) \mapsto f_p(x)$ is.

**Proposition 6** ([Kva25a])**.** Let $P$ be a compact smooth manifold with boundary and $\{f_p\}_{p\in\partial P}$ be a $C^k$ family of vector fields $f_p \in \mathcal{S}^k(\mathbb{R}^n)$. Then there is a $C^k$ family $\{g_p\}_{p\in P}$ of vector fields $g_p \in \mathcal{S}^k(\mathbb{R}^n)$ extending $\{f_p\}_{p\in\partial P}$, provided either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 3$.

This proposition is proved as follows. By Corollary 1 and homotopy theory, there is a continuous family $\{h_p\}_{p\in P}$ of vector fields $h_p \in \mathcal{S}^k(\mathbb{R}^n)$ extending $\{f_p\}_{p\in\partial P}$ [Hat02, Lem. 4.7]. This continuous family is then upgraded to the desired $C^k$ family $\{g_p\}_{p\in P}$ using smoothing methods.

5.1.2. *Conley's question and smooth continuation.* Taking the parameter space $P = [0,1]$ in Proposition 6 gives a smooth version of the partial answers to Conley's continuation question [Con78, Section IV.8.1.A] provided by Jongeneel [Jon26, Theorem 3.6] and Proposition 5 (Figure 3). Taking $P$ to be of the more general form $[0,1] \times Q$ then yields a further parametric extension of this partial answer.

These results are stated in terms of global asymptotic stability for convenience in the present paper, but they can also be converted into statements about continuation and boundary value problems involving local asymptotic stability [Kva25a].

Using techniques from Smale's proof of the $h$-cobordism theorem [Sma62], Reineck proved a theorem giving another smooth partial answer to Conley's question [Rei91, Rei92]. Using Cerf theory [Cer70], Mischaikow and Parikh have recently obtained one-parameter extensions of some of Reineck's results [MP26]. However, in

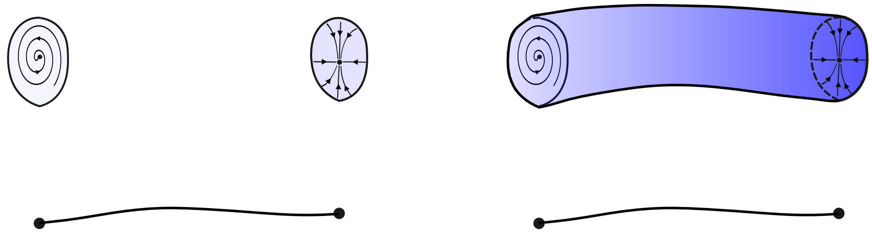

Figure 3. Continuation of asymptotically stable vector fields.

our context, the continuations in these references may encounter isolated invariant sets more complicated than equilibria, which we aim to avoid here.

5.1.3. *Parametric Hartman–Grobman theorem without hyperbolicity.* Proposition 6 can be used to give a short proof of the following parametric extension of Theorem 3.

**Theorem 5** ([Kva25a])**.** Let $P$ be a compact smooth manifold with boundary and $\{L_p\}_{p\in P}$ be a smooth family of proper strict Lyapunov functions $L_p\colon \mathbb{R}^n \to [0,\infty)$ for some continuous family of asymptotically stable vector fields. Given $a>0$, let

$$N_a = \bigcup_{p\in P} \{p\} \times L_p^{-1}([0,a]) \subset P \times \mathbb{R}^n.$$

Fix $b, c > 0$. Then there is a diffeomorphism $F\colon N_b \to N_c$ satisfying $\mathrm{Pr}_1 \circ F = \mathrm{Pr}_1|_{N_b}$. Moreover, there is a diffeomorphism $G\colon N_c \to P \times D^n$ satisfying $\mathrm{Pr}_1 \circ G = \mathrm{Pr}_1|_{N_c}$ provided either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 2$.

In the same way that Theorem 3 led to Proposition 4, the above theorem leads to the following parametric extension of the latter result, a parametric Hartman–Grobman theorem for nonhyperbolic but asymptotically stable equilibria.

**Proposition 7** ([Kva25a])**.** Let $P$ be a compact smooth manifold with boundary and $\{f_p\}_{p\in P}$ be a continuous family of complete asymptotically stable vector fields $f_p$ on $\mathbb{R}^n$. Then if either (i) $n < 4$ or (ii) $n > 5$ and $\dim P < 2$, there is a continuous family $\{H_p\}_{p\in P}$ of homeomorphisms $H_p\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$H_p(x_p(t)) = e^{-t} H_p(x_p(0)) \tag{6}$$

for all solutions of $\dot{x}_p = f_p(x_p)$, $p \in P$. Moreover, if $\{f_p\}_{p\in P}$ is a $C^k$ family with $k \in \mathbb{N} \cup \{\infty\}$, then $\{H_p\}_{p\in P}$ may be chosen to restrict to a $C^k$ family of diffeomorphisms on the complement of the set of equilibria.

**Remark 12.** While this proposition is a parametric extension of Proposition 4, the latter has fewer dimensional restrictions.

**Remark 13.** As in Remark 5, for the second statement of Proposition 7, the proof requires only that $\{f_p\}_{p\in P}$ generates a family of flows that is $C^k$ on the complement of the set of equilibria.

**Remark 14.** As in Remark 6, techniques similar to those in the proof of Proposition 7 can be used to prove a parametric extension of the Morse lemma for local minima of non-Morse functions due to Grüne, Wirth, and Sontag [GSW99, Proposition 1] (see [Kva25a]).

5.2. **Proof sketch.** In this section, we sketch the proof of Theorem 4 and deduce Corollary 1. Theorem 4 is proved by constructing a pair of weak homotopy equivalences from a suitable space of Lyapunov functions to $\mathcal{S}^k(\mathbb{R}^n)$ and $\mathrm{Gr}(D^n, \mathbb{R}^n)$.

5.2.1. *Reduction to the space of Lyapunov functions.* Let $\mathcal{L}(\mathbb{R}^n)$ denote the space of proper surjective smooth functions $\mathbb{R}^n \to [0, \infty)$ having a unique critical point, which is necessarily the unique point attaining the global minimum 0. Let $\mathcal{L}_0(\mathbb{R}^n)$ denote the subspace of functions equal to 0 at the origin. We equip both $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{L}_0(\mathbb{R}^n)$ with the $C^\infty$ topology.

**Remark 15.** By Theorem 1 and consideration of gradient flows, $\mathcal{L}(\mathbb{R}^n)$ coincides with the space of all smooth proper strict Lyapunov functions for asymptotically stable vector fields on $\mathbb{R}^n$. Similarly, $\mathcal{L}_0(\mathbb{R}^n)$ coincides with the subspace of such Lyapunov functions for which the equilibrium of the vector field is at the origin.

**Theorem 6** ([Kva25a])**.** There is a commutative diagram of embeddings

$$\begin{array}{ccc} \mathcal{L}(\mathbb{R}^n) & \xhookrightarrow{-\nabla} & \mathcal{S}^k(\mathbb{R}^n) \\ \uparrow & & \uparrow \\ \mathcal{L}_0(\mathbb{R}^n) & \xhookrightarrow{-\nabla} & \mathcal{S}^k_0(\mathbb{R}^n) \end{array}$$

in which the "take the negative gradient" maps are weak homotopy equivalences, and the vertical maps are homotopy equivalences given by subspace inclusion.

A sketch of a proof follows.

First, parametrically translating functions yields strong deformation retractions of $\mathcal{L}(\mathbb{R}^n)$ and $\mathcal{S}^k(\mathbb{R}^n)$ onto $\mathcal{L}_0(\mathbb{R}^n)$ and $\mathcal{S}^k_0(\mathbb{R}^n)$, respectively, implying that the vertical maps are homotopy equivalences.

Next, to prove that the horizontal maps are weak homotopy equivalences, one must consider a continuous family $\{f_p\}_{p \in D^m}$ of vector fields $f_p \in \mathcal{S}^k(\mathbb{R}^n)$ satisfying $f_p \in -\nabla\mathcal{L}(\mathbb{R}^n)$ for $p \in \partial D^m$ and construct a homotopy through such families, fixed on $\partial D^m$, to a family $\{g_p\}_{p \in D^m}$ satisfying $g_p \in -\nabla\mathcal{L}(\mathbb{R}^n)$ for all $p \in D^m$ [Hat02, p. 343].

The family $\{g_p\}_{p \in D^m}$ can be constructed as the negative gradient of a suitably adjusted family of functions provided by Wilson's converse Lyapunov function theorem [Wil69, Theorem 3.2], [FP19, Section 6], which is a generalization of Theorem 1 to asymptotically stable compact invariant sets beyond equilibria. The homotopy between the families can then be chosen to move in straight lines.

5.2.2. *Reduction to the nonlinear Grassmannian of discs.* In the proof of Theorem 3, we used asymptotically stable vector fields to study sublevel sets of Lyapunov functions. Together with Theorem 6, the next result will allow us to turn this around: we will use the space of Lyapunov function sublevel sets to study the space of asymptotically stable vector fields.

Let $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ be the open subspace of the nonlinear Grassmannian $\mathrm{Gr}(D^n, \mathbb{R}^n)$ consisting of neighborhoods of $0 \in \mathbb{R}^n$. By Theorem 3, the 1-sublevel set $L^{-1}([0,1])$ of any $L \in \mathcal{L}_0(\mathbb{R}^n)$ is diffeomorphic to $D^n$ provided $n \neq 4, 5$, so it can be viewed as a point in $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$. Thus, the following statement makes sense. See Figure 4.

**Theorem 7** ([Kva25a])**.** If $n \neq 4, 5$, the **sublevel set map** defined by

$$q\colon \mathcal{L}_0(\mathbb{R}^n) \to \mathrm{Gr}_0(D^n, \mathbb{R}^n), \quad q(L) = L^{-1}([0,1])$$

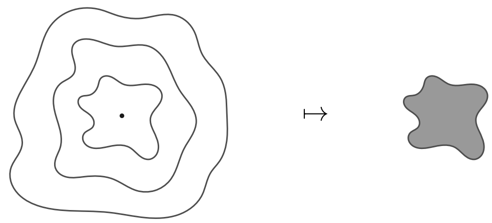

FIGURE 4. Level sets of a function in $\mathcal{L}_0(\mathbb{R}^2)$, and the image of this function under the sublevel set map (see Theorem 7).

is a surjective and locally trivial fiber bundle with weakly contractible fibers. In particular, it is a weak homotopy equivalence.

**Remark 16.** It is straightforward to show that the inclusion $\mathrm{Gr}_0(D^n, \mathbb{R}^n) \hookrightarrow \mathrm{Gr}(D^n, \mathbb{R}^n)$ is a weak homotopy equivalence. Thus, Theorems 6 and 7 imply that $\mathcal{S}^k(\mathbb{R}^n)$ has the same weak homotopy type as $\mathrm{Gr}(D^n, \mathbb{R}^n)$, establishing Theorem 4.

A sketch of a proof of Theorem 7 follows.

First, a theorem of Abraham on smoothness of evaluation maps [AR67, Theorem 10.3] and the implicit function theorem of Hildebrandt and Graves [HG27, Theorem 4] imply that $q$ is continuous. The disc theorem (Lemma 2) then implies that $q$ is surjective and $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is path-connected.

Next, to show that $q$ is a (locally trivial) fiber bundle, we need to construct a local trivialization over a neighborhood of an arbitrary $M \in \mathrm{Gr}_0(D^n, \mathbb{R}^n)$. Given $M$, we can use a tubular neighborhood of its boundary to construct a neighborhood $U \subset \mathrm{Gr}_0(D^n, \mathbb{R}^n)$ and a continuous map $\Psi\colon U \to \mathrm{Diff}(\mathbb{R}^n)$ satisfying $\Psi(N)(M) = N$ and $\Psi(N)(0) = 0$ for all $N \in U$. We now define the fiber $\mathcal{F}_M = q^{-1}(M)$ and map $F\colon q^{-1}(U) \to \mathcal{F}_M$ by

$$F(L) = L \circ \Psi(q(L)).$$

The map $(q, F)\colon q^{-1}(U) \to U \times \mathcal{F}_M$ is continuous, and it is a homeomorphism since it has a continuous inverse $G\colon U \times \mathcal{F}_M \to q^{-1}(U)$ given by $G(N, J) = J \circ \Psi(N)^{-1}$. Thus, $q$ is a fiber bundle.

To show that $q$ has weakly contractible fibers, since $\mathrm{Gr}_0(D^n, \mathbb{R}^n)$ is path-connected, it suffices to show that $\mathcal{F} := \mathcal{F}_{D^n} = q^{-1}(D^n)$ is weakly contractible. Note that $\mathcal{F}$ is just the set of functions in $\mathcal{L}_0(\mathbb{R}^n)$ whose 1-sublevel set is the standard $n$-disc. Thus, it is intuitive that there is a homotopy from any given continuous family $\{L_p\}_{p \in S^m}$ of functions $L_p \in \mathcal{L}_0(\mathbb{R}^n)$ through such families to the constant family of functions $x \mapsto |x|^2$ by "parting the sea" of level sets away from $\partial D^n = S^{n-1}$, replacing the sea with level sets of $x \mapsto |x|^2$, and composing with a suitably "flat" function to smooth any singularities arising at the origin. See Figure 5.

Since $q$ is a fiber bundle with weakly contractible fibers, it is a weak homotopy equivalence by the long exact sequence of homotopy groups of a fibration [Hat02, Theorem 4.41].

5.2.3. *Homotopy groups of the nonlinear Grassmannian.* Corollary 1 is a consequence of Theorem 4 together with the following result. The proof applies results of Smale, Cerf, and Hatcher [Sma59, Cer70, Hat83] using a result of Gay-Balmaz and Vizman [GBV14] and techniques from the study of embedding spaces [ABK72, Bud08, CMRW17, Kup19].

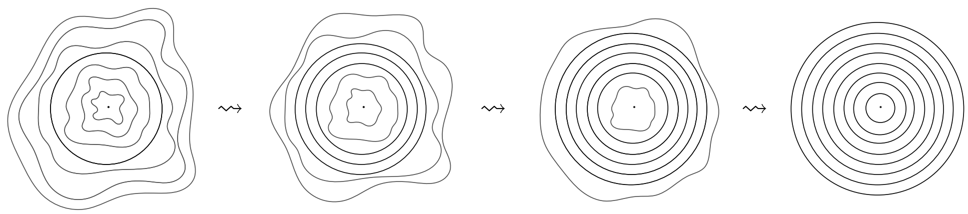

FIGURE 5. Schematic homotopy in the standard fiber $\mathcal{F} = \{L \in \mathcal{L}_0(\mathbb{R}^n) \colon L^{-1}([0,1]) = D^n\}$. This can be done parametrically to establish weak contractibility of $\mathcal{F}$, as in the proof of Theorem 7.

**Lemma 5.** $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is path-connected for all $n$, simply connected if $n \neq 4, 5$, and weakly contractible if $n \leq 3$.

*Proof.* As noted in the proof of Theorem 7, the disc theorem of Palais and Cerf implies that $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is path-connected for all $n$. We now prove the remaining statements using results of Smale, Cerf, and Hatcher [Sma59, Cer70, Hat83].

By generalizing a theorem of Binz and Fischer [BF81] based on an idea implicit in work of Weinstein [Wei71], Gay-Balmaz and Vizman proved that the natural quotient map

$$\mathrm{Emb}^+(D^n, \mathbb{R}^n) \to \mathrm{Gr}(D^n, \mathbb{R}^n), \quad f \mapsto f(D^n)$$

is a principal $\mathrm{Diff}^+(D^n)$-bundle, hence also a Serre fibration, where $\mathrm{Emb}^+$ and $\mathrm{Diff}^+$ denote orientation-preserving embeddings and diffeomorphisms [GBV14]. Thus, there is a long exact sequence of homotopy groups [Hat02, Theorem 4.41]

$$\cdots \pi_k \mathrm{Diff}^+(D^n) \longrightarrow \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \longrightarrow \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \cdots$$

containing the segment

$$\begin{array}{c} \pi_k \mathrm{Diff}^+(D^n) \xrightarrow{\text{surjective}} \pi_k \mathrm{Emb}^+(D^n, \mathbb{R}^n) \xrightarrow{\;0\;} \pi_k \mathrm{Gr}(D^n, \mathbb{R}^n) \\ \xhookrightarrow{\text{injective}} \pi_{k-1} \mathrm{Diff}^+(D^n) \xrightarrow{\text{surjective}} \pi_{k-1} \mathrm{Emb}^+(D^n, \mathbb{R}^n) \end{array} \tag{7}$$

To see that the indicated maps are indeed surjective, note that they are induced by the top inclusion in the commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}^+(D^n) & \hookrightarrow & \mathrm{Emb}^+(D^n, \mathbb{R}^n) \\ \uparrow & \searrow & \downarrow \simeq \\ SO(n) & \xhookrightarrow[\simeq]{} & GL^+(n) \end{array} \tag{8}$$

in which the diagonal and right vertical maps are "evaluate the derivative at a point". The bottom inclusion is a homotopy equivalence by the Gram-Schmidt process, so the diagonal map is surjective on homotopy groups. And the right vertical map is a homotopy equivalence by "zooming in" at the origin, so the top inclusion is indeed surjective on homotopy groups. Exactness then implies that the other maps in (7) are 0 or injective where indicated.

Cerf's pseudo-isotopy theorem implies that $\pi_0 \mathrm{Diff}^+(D^n)$ is trivial for $n > 5$ [Cer70], so injectivity of the long map in (7) implies that $\pi_1 \mathrm{Gr}(D^n, \mathbb{R}^n)$ is trivial and hence $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is simply connected for $n > 5$.

It remains only to show that $\mathrm{Gr}(D^n, \mathbb{R}^n)$ is weakly contractible if $n < 4$. By exactness, it suffices to show that the surjections in (7) become bijections if $n < 4$. Since those maps are induced by the top inclusion in (8), it suffices to show that the diagonal map in (8) is a weak homotopy equivalence if $n < 4$.

Let $\mathrm{Fr}^+(TS^{n-1})$ be the bundle of positively oriented $(n-1)$-frames of vectors tangent to $S^{n-1}$. The diagonal map in (8) is homotopic to the composition

$$\mathrm{Diff}^+(D^n) \xrightarrow{\rho} \mathrm{Diff}^+(S^{n-1}) \xrightarrow{F} \mathrm{Fr}^+(TS^{n-1}) \xhookrightarrow{\simeq} GL^+(n)$$

of the restriction map $\rho$, the map $F$ given by adjoining the value and derivative at the north pole of $S^{n-1}$, and the inclusion map. The inclusion map is a homotopy equivalence by the first step of the Gram-Schmidt process, so we need only prove that $\rho$ and $F$ are weak homotopy equivalences if $n < 4$.

Cerf's first fibration theorem implies that $\rho$ is a fiber bundle [Cer61, Section II.2.2.2, Corollary 2], and the fiber over $\mathrm{id}_{S^{n-1}}$ is

$$\mathrm{Diff}_\partial(D^n) \coloneqq \{\text{diffeomorphisms of } D^n \text{ that are the identity on } \partial D^n\}.$$

This fiber is contractible for $n = 1$ by a convexity argument, for $n = 2$ by a theorem of Smale [Sma59], and for $n = 3$ by Hatcher's proof of the Smale conjecture [Hat83]. Thus, $\rho$ is a weak homotopy equivalence for $n < 4$ by the long exact sequence of homotopy groups of a fibration [Hat02, Theorem 4.41].

To complete the proof, we need to show that $F$ is a weak homotopy equivalence for $n < 4$. This is trivial for $n = 1$, so we need only consider $1 < n < 4$. Observe that $F$ factors as the composition

$$\mathrm{Diff}^+(S^{n-1}) \longrightarrow \mathrm{Emb}^+(D_+^{n-1}, S^{n-1}) \xrightarrow{\simeq} \mathrm{Emb}^+(\mathrm{Int}\, D_+^{n-1}, S^{n-1}) \xhookrightarrow{\simeq} \mathrm{Fr}^+(TS^{n-1})$$

in which $D_+^{n-1}$ is the upper hemisphere of $S^{n-1}$, the first two maps are restrictions, and the long map adjoins the value and derivative at the north pole. The long map is well-known to be a weak homotopy equivalence [Cer61, Section II.5.1.5, Corollary]. The second restriction is a homotopy equivalence since it is given by composition with the inclusion map $\mathrm{Int}\, D_+^{n-1} \hookrightarrow D_+^{n-1}$, which has an inverse up to isotopy corresponding to the linear embedding $D^{n-1} \hookrightarrow \mathrm{Int}\, D^{n-1}$, $x \mapsto \frac{1}{2}x$. So we need only show that the first restriction is a weak homotopy equivalence. That restriction is a fiber bundle by Cerf's first fibration theorem [Cer61, Section II.2.2.2, Corollary 2], so by the long exact sequence of homotopy groups [Hat02, Theorem 4.41] it suffices to observe that its fiber

$$\mathrm{Diff}(S^{n-1} \text{ rel } D_+^{n-1}) \simeq \mathrm{Diff}_\partial(D^{n-1})$$

over the inclusion map is weakly contractible by the aforementioned convexity argument for $n = 2$ and theorem of Smale for $n = 3$ [Sma59]. This proves that $F$ and hence also the diagonal map in (8) are weak homotopy equivalences for $n < 4$. □

## Acknowledgments

This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-24-1-0299. The author is grateful to Indika Rajapakse and Stephen Smale for inviting him to present the work in Section 5 at the Smale@95 conference hosted by the Simons Institute in Berkeley, CA. The author would like to thank Daniel E. Koditschek and Eduardo D. Sontag for collaborations related to Propositions 3 and 4, Wouter Jongeneel for useful discussions regarding [Jon26], and Ryan Budney, Alexander Kupers, Mike Miller Eismeier, Richard Montgomery, and Álvaro del Pino Gómez for helpful conversations regarding Section 5.

## References

[ABK72] P. L. Antonelli, D. Burghelea, and P. J. Kahn, *The non-finite homotopy type of some diffeomorphism groups*, Topology **11** (1972), 1–49. MR 292106

[Abr63] R. Abraham, *Lectures of Smale on differential topology*, 1962–1963, Notes from lectures by S. Smale, Columbia University.

[AR67] R. Abraham and J. Robbin, *Transversal mappings and flows*, W. A. Benjamin, Inc., New York-Amsterdam, 1967, An appendix by Al Kelley. MR 240836

[BB00] S. P. Bhat and D. S. Bernstein, *A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon*, Systems Control Lett. **39** (2000), no. 1, 63–70. MR 1828234

[BF81] E. Binz and H. R. Fischer, *The manifold of embeddings of a closed manifold*, Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978), Lecture Notes in Phys., vol. 139, Springer, Berlin-New York, 1981, With an appendix by P. Michor, pp. 310–329. MR 613004

[BK74] N. A. Bobylev and M. A. Krasnosel'skiĭ, *Deformation of a system into an asymptotically stable system*, Automation and Remote Control **35** (1974), no. 7, 1041–1044.

[BK25] M.-A. Belabbas and J. Ko, *Dynamic global feedback stabilization: why do the twist?*, Geometry, topology and control system design, AIMS Ser. Appl. Math., vol. 13, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2025, pp. 47–60. MR 5010887

[Bro60] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76. MR 117695

[Bro61] ———, *The monotone union of open $n$-cells is an open $n$-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814. MR 126835

[Bro62] ———, *Locally flat imbeddings of topological manifolds*, Ann. of Math. (2) **75** (1962), 331–341. MR 133812

[Bro83] R. W. Brockett, *Asymptotic stability and feedback stabilization*, Differential geometric control theory (Houghton, Mich., 1982), Progr. Math., vol. 27, Birkhäuser Boston, Boston, MA, 1983, pp. 181–191. MR 708502

[BS70] N. P. Bhatia and G. P. Szegö, *Stability theory of dynamical systems*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 161, Springer-Verlag, New York-Berlin, 1970. MR 289890

[Bud08] R. Budney, *A family of embedding spaces*, Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, pp. 41–83. MR 2508201

[BV02] C. Bonatti and J. Villadelprat, *The index of stable critical points*, Topology Appl. **126** (2002), no. 1-2, 263–271. MR 1934263

[Byr08] C. I. Byrnes, *On Brockett's necessary condition for stabilizability and the topology of Liapunov functions on* $\mathbb{R}^n$, Communications in Information and Systems **8** (2008), no. 4, 333–352.

[Cer61] J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380. MR 140120

[Cer70] ———, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 39, 5–173. MR 292089

[Che61] N. G. Chetayev, *The stability of motion*, Pergamon Press, Oxford-New York-Paris, 1961, Translated from the Russian by M. Nadler. MR 271487

[CLS98] F. H. Clarke, Yu. S. Ledyaev, and R. J. Stern, *Asymptotic stability and smooth Lyapunov functions*, J. Differential Equations **149** (1998), no. 1, 69–114. MR 1643670

[CLSS97] F. H. Clarke, Yu. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control **42** (1997), no. 10, 1394–1407. MR 1472857

[CMRW17] F. Cantero Morán and O. Randal-Williams, *Homological stability for spaces of embedded surfaces*, Geom. Topol. **21** (2017), no. 3, 1387–1467. MR 3650077

[Col65] C. Coleman, *Local trajectory equivalence of differential systems*, Proc. Amer. Math. Soc. **16** (1965), 890–892. MR 180752

[Col66] ———, *Addendum to: Local trajectory equivalence of differential systems*, Proc. Amer. Math. Soc. **17** (1966), 770. MR 192135

[Con78] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 1978. MR 511133

[Cor90] J.-M. Coron, *A necessary condition for feedback stabilization*, Systems & Control Letters **14** (1990), no. 3, 227–232.

[Cor92] ———, *Global asymptotic stabilization for controllable systems without drift*, Math. Control Signals Systems **5** (1992), no. 3, 295–312. MR 1164379

[Cor07] ———, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007. MR 2302744

[Edw84] R. D. Edwards, *The solution of the* 4*-dimensional annulus conjecture (after Frank Quinn)*, Four-manifold theory (Durham, N.H., 1982), Contemp. Math., vol. 35, Amer. Math. Soc., Providence, RI, 1984, pp. 211–264. MR 780581

[Fat22] A. Fathi, *On the existence of smooth Lyapunov functions for arbitrary closed sets*, Bull. Braz. Math. Soc. (N.S.) **53** (2022), no. 3, 957–980. MR 4458796

[FP19] A. Fathi and P. Pageault, *Smoothing Lyapunov functions*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 1677–1700. MR 3894031

[Fre82] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982), no. 3, 357–453. MR 679066

[GBV14] F. Gay-Balmaz and C. Vizman, *Principal bundles of embeddings and nonlinear Grassmannians*, Ann. Global Anal. Geom. **46** (2014), no. 3, 293–312. MR 3263203

[Gro59] D. M. Grobman, *Homeomorphism of systems of differential equations*, Doklady Akademii Nauk SSSR **128** (1959), no. 5, 880–881.

[GSW99] L. Grüne, E. D. Sontag, and F. R. Wirth, *Asymptotic stability equals exponential stability, and ISS equals finite energy gain—if you twist your eyes*, Systems Control Lett. **38** (1999), no. 2, 127–134. MR 1751688

[Har60] P. Hartman, *A lemma in the theory of structural stability of differential equations*, Proc. Amer. Math. Soc. **11** (1960), 610–620. MR 121542

[Har82] ———, *Ordinary differential equations*, second ed., Birkhäuser, Boston, MA, 1982. MR 658490

[Har83] D. Hart, *On the smoothness of generators*, Topology **22** (1983), no. 3, 357–363. MR 710109

[Hat83] A. E. Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. (2) **117** (1983), no. 3, 553–607. MR 701256

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[HG27] T. H. Hildebrandt and L. M. Graves, *Implicit functions and their differentials in general analysis*, Trans. Amer. Math. Soc. **29** (1927), no. 1, 127–153. MR 1501380

[Hir65] M. W. Hirsch, *On homotopy spheres of low dimension*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press, Princeton, NJ, 1965, pp. 199–204. MR 179800

[Hir94] ———, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original. MR 1336822

[Hop27] H. Hopf, *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*, Math. Ann. **96** (1927), no. 1, 225–249. MR 1512316

[HS74] M. W. Hirsch and S. Smale, *Differential equations, dynamical systems, and linear algebra*, Pure and Applied Mathematics, vol. 60, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1974. MR 486784

[JM23] W. Jongeneel and E. Moulay, *Topological obstructions to stability and stabilization—history, recent advances and open problems*, SpringerBriefs in Electrical and Computer Engineering, Springer, Cham, 2023, SpringerBriefs in Control, Automation and Robotics. MR 4633311

[Jon26] W. Jongeneel, *Asymptotic stability equals exponential stability*—while *you twist your eyes*, Systems Control Lett. **208** (2026), Paper No. 106324, 9. MR 5002245

[Kel15] C. M. Kellett, *Classical converse theorems in Lyapunov's second method*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), no. 8, 2333–2360. MR 3423238

[Kir69] R. C. Kirby, *Stable homeomorphisms and the annulus conjecture*, Ann. of Math. (2) **89** (1969), 575–582. MR 242165

[KK22] M. D. Kvalheim and D. E. Koditschek, *Necessary conditions for feedback stabilization and safety*, J. Geom. Mech. **14** (2022), no. 4, 659–693. MR 4484131

[Kos93] A. A. Kosinski, *Differential manifolds*, Pure and Applied Mathematics, vol. 138, Academic Press, Inc., Boston, MA, 1993. MR 1190010

[Kra63] N. N. Krasovskiĭ, *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay*, Stanford University Press, Stanford, CA, 1963, Translated by J. L. Brenner. MR 147744

[KS25] M. D. Kvalheim and E. D. Sontag, *Global linearization of asymptotically stable systems without hyperbolicity*, Systems Control Lett. **203** (2025), Paper No. 106163. MR 4919921

[Kup19] A. Kupers, *Lectures on diffeomorphism groups of manifolds*, Lecture notes, available at webpage https://www.utsc.utoronto.ca/people/kupers/teaching/diffeomorphism-groups-lecture-notes, 2019.

[Kur56] J. Kurzweil, *On the inversion of Lyapunov's second theorem on stability of motion*, Czechoslovak Mathematical Journal **81** (1956), 217–259, 455–484, English translation in Amer. Math. Soc. Transl. (2), **24**, 19–77.

[Kva23] M. D. Kvalheim, *Obstructions to asymptotic stabilization*, SIAM J. Control Optim. **61** (2023), no. 2, 536–542. MR 4573255

[Kva25a] ______, *Differential topology of the spaces of asymptotically stable vector fields and Lyapunov functions*, arXiv preprint arXiv:2503.10828 (2025), 1–35.

[Kva25b] ______, *Relationships between necessary conditions for feedback stabilizability*, Geometry, topology and control system design, AIMS Ser. Appl. Math., vol. 13, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2025, pp. 167–179. MR 5010894

[KZ84] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical methods of nonlinear analysis*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 263, Springer-Verlag, Berlin, 1984, Translated from the Russian by C. C. Fenske. MR 736839

[LaS60] J. P. LaSalle, *The extent of asymptotic stability*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 363–365. MR 113014

[Lee13] J. M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR 2954043

[Lya92] A. M. Lyapunov, *The general problem of the stability of motion*, Internat. J. Control **55** (1992), no. 3, 521–790, Translated by A. T. Fuller from É. Davaux's French translation (1907) of the 1892 Russian original, With an editorial (historical introduction) by Fuller, a biography of Lyapunov by V. I. Smirnov, and the bibliography of Lyapunov's works collected by J. F. Barrett, Lyapunov centenary issue. MR 1154209

[Man07] A.-R. Mansouri, *Local asymptotic feedback stabilization to a submanifold: topological conditions*, Systems Control Lett. **56** (2007), no. 7-8, 525–528. MR 2332004

[Mas56] J. L. Massera, *Contributions to stability theory*, Ann. of Math. (2) **64** (1956), 182–206. MR 79179

[Maz59] B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65** (1959), 59–65. MR 117693

[Mil63] J. Milnor, *Morse theory*, Annals of Mathematics Studies, vol. No. 51, Princeton University Press, Princeton, NJ, 1963, Based on lecture notes by M. Spivak and R. Wells. MR 163331

[Mil64] ______, *Differential topology*, Lectures on Modern Mathematics, Vol. II, Wiley, New York-London-Sydney, 1964, pp. 165–183. MR 178474

[Mil65] ———, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, NJ, 1965, Notes by L. Siebenmann and J. Sondow. MR 190942

[Mil97] J. W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by D. W. Weaver, Revised reprint of the 1965 original. MR 1487640

[Mor96] M. Morse, *The calculus of variations in the large*, American Mathematical Society Colloquium Publications, vol. 18, American Mathematical Society, Providence, RI, 1996, Reprint of the 1932 original. MR 1451874

[MP26] K. Mischaikow and A. Parikh, *Realizing saddle-node bifurcations from finite data*, arXiv preprint arXiv:2605.02068 (2026), 1–12.

[MT07] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. MR 2334563

[Mun66] J. R. Munkres, *Elementary differential topology*, revised ed., Annals of Mathematics Studies, vol. No. 54, Princeton University Press, Princeton, NJ, 1966, Lectures given at Massachusetts Institute of Technology, Fall, 1961. MR 198479

[Pal60] R. S. Palais, *Extending diffeomorphisms*, Proc. Amer. Math. Soc. **11** (1960), 274–277. MR 117741

[Per02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv preprint math/0211159 (2002), 1–39.

[Per03a] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv preprint math/0307245 (2003), 1–7.

[Per03b] ———, *Ricci flow with surgery on three-manifolds*, arXiv preprint math/0303109 (2003), 1–22.

[Poi85] H. Poincaré, *Sur les courbes définies par les équations différentielles*, Journal de Mathématiques Pures et Appliquées **1** (1885), 167–244 (French).

[Pug68] C. C. Pugh, *A generalized Poincaré index formula*, Topology **7** (1968), 217–226. MR 229254

[Qui82] F. Quinn, *Ends of maps. III. Dimensions* 4 *and* 5, J. Differential Geometry **17** (1982), no. 3, 503–521. MR 679069

[Rei91] J. F. Reineck, *Continuation to gradient flows*, Duke Math. J. **64** (1991), no. 2, 261–269. MR 1136376

[Rei92] ———, *Continuation to the minimal number of critical points in gradient flows*, Duke Math. J. **68** (1992), no. 1, 185–194. MR 1185822

[Sma59] S. Smale, *Diffeomorphisms of the* 2*-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626. MR 112149

[Sma61] ———, *Generalized Poincaré conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406. MR 137124

[Sma62] ———, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399. MR 153022

[Son98] E. D. Sontag, *Mathematical control theory*, second ed., Texts in Applied Mathematics, vol. 6, Springer-Verlag, New York, 1998, Deterministic finite-dimensional systems. MR 1640001

[Son99] ———, *Nonlinear feedback stabilization revisited*, Dynamical systems, control, coding, computer vision (Padova, 1998), Progr. Systems Control Theory, vol. 25, Birkhäuser, Basel, 1999, pp. 223–262. MR 1684843

[Sta62] J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488. MR 149457

[Wal16] C. T. C. Wall, *Differential topology*, Cambridge Studies in Advanced Mathematics, vol. 156, Cambridge University Press, Cambridge, 2016. MR 3558600

[Wei71] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Math. **6** (1971), 329–346. MR 286137

[Wil67] F. W. Wilson, Jr., *The structure of the level surfaces of a Lyapunov function*, J. Differential Equations **3** (1967), 323–329. MR 0231409

[Wil69] ———, *Smoothing derivatives of functions and applications*, Trans. Amer. Math. Soc. **139** (1969), 413–428. MR 0251747

[Wil78] ______, *A reformulation of Coleman's conjecture concerning the local conjugacy of topologically hyperbolic singular points*, The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), Lecture Notes in Math., vol. 668, Springer, Berlin-New York, 1978, pp. 245–252. MR 518564

[Zab89] J. Zabczyk, *Some comments on stabilizability*, Appl. Math. Optim. **19** (1989), no. 1, 1–9. MR 955087

[Zab20] ______, *Mathematical control theory: an introduction*, second ed., Systems & Control: Foundations & Applications, Birkhäuser/Springer, Cham, 2020. MR 4211767

Department of Mathematics and Statistics, University of Maryland, Baltimore County, MD, USA

*Email address*: `kvalheim@umbc.edu`