

# Discovering engineering (im)possibilities with geometry and topology<sup>1</sup>

Matthew D. Kvalheim

Department of Mathematics and Statistics  
University of Maryland, Baltimore County

*kvalheim@umbc.edu*

October 25, 2023

---

<sup>1</sup>Funding by the Army Research Office (MURI W911NF-18-1-0327) and the Office of Naval Research (VBF N00014-16-1-2817) is gratefully acknowledged.

# Discovering engineering (im)possibilities for:

Feedback stabilizability

Applied Koopman operator methods

Deep neural network autoencoders

# Discovering engineering (im)possibilities for:

## Feedback stabilizability

- Brockett's necessary condition and beyond

- A homotopy theorem beyond the Coron/Mansouri tests

- Periodic orbits can be easier to stabilize than equilibria

## Applied Koopman operator methods

## Deep neural network autoencoders

## Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \quad (1)$$

where  $M \ni x$  is a smooth manifold and  $f$  is smooth.

1. **Controllability problem:** Given  $a, b \in M$ , find  $u(t)$  s.t.  $x(T) = b$  if  $x(0) = a$  for some  $T > 0$ .

$$a \rightsquigarrow b$$

2. **Stabilizability problem:** Given a compact subset  $A \subset M$ , find smooth  $u(x)$  s.t.  $A$  is **asymptotically stable**<sup>2</sup> for the **closed-loop vector field**  $F(x) = f(x, u(x))$ . [▶ Link](#)

---

<sup>2</sup>For every open  $W \supset A$  there is an open  $V \supset A$  s.t. all forward  $F$ -trajectories initialized in  $V$  are contained in  $W$  and converge to  $A$ .

## The stabilization conjecture and Brockett's solution

Often  $A = \{x_*\}$  is a point,  $M = \mathbb{R}^n$  in the stabilization problem.

**Stabilization conjecture (pre-1983):** a reasonably strong form of controllability implies smooth stabilizability of a point.

**Example:** the “Heisenberg system” or “nonholonomic integrator”

$$\left. \begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?

**Theorem (Brockett).** If a point is stabilizable, then  $\text{image}(f)$  is a neighborhood of 0. (In the example,  $(0, 0, \varepsilon) \notin \text{image}(f)$ .)

## Other stabilizability work

- ▶ Exponential (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
- ▶ global (Byrnes 2008, Baryshnikov 2023),
- ▶ time-varying (Coron 1992), and
- ▶ discontinuous (Clarke, Ledyaev, Sontag, Subbotin 1997)

variants of the stabilization problem are not considered here.

## Coron's and Mansouri's obstructions

Krasnosel'skiĭ and Zabreĭko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : f(x, u) \neq 0\}.$$

**Theorem (Coron 1990).** If  $n > 1$  and a point is stabilizable,

$$f_*(H_{n-1}(\Sigma)) = H_{n-1}(\mathbb{R}^n \setminus \{0\}) \quad (\cong \mathbb{Z}).$$

**Theorem (Mansouri 2010).** If a closed codimension  $> 1$  submanifold  $A \subset \mathbb{R}^n$  with Euler characteristic  $\chi(A)$  is stabilizable,

$$f_*(H_{n-1}(\Sigma)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \quad (\cong \chi(A) \cdot \mathbb{Z}).$$

## Limitations of these results

The results of Brockett, Coron, Mansouri rely on parallelizability of  $\mathbb{R}^n$  to view vector fields and control systems as  $\mathbb{R}^n$ -valued.

Furthermore, they apply only to the special case that  $A$  is a point or a closed submanifold of  $\mathbb{R}^n$  with  $\chi(A) \neq 0$ .

But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

How to test for stabilizability in such general settings?<sup>3</sup>

- ▶ **Generalization of Brockett's test** (MDK and Daniel E. Koditschek, J Geometric Mechanics, 2022).
- ▶ **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).

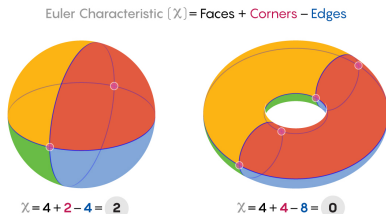
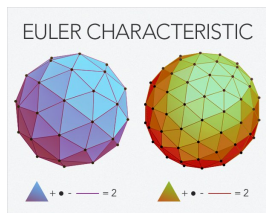
---

<sup>3</sup>An exposition of all stabilizability results here is in 2023 book *Topological Obstructions to Stability and Stabilization* by W. Jongeneel and E. Moulay.



# A primer on the Euler characteristic<sup>4</sup>

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).



**Notation:**  $\chi(Y) :=$  Euler characteristic of  $Y$ .

**Examples:**  $\chi(\bullet) = 1$ ,  $\chi(S^1) = 0$ ,  $\chi(S^2) = 2$ ,  $\chi(\text{figure 8}) = -1$

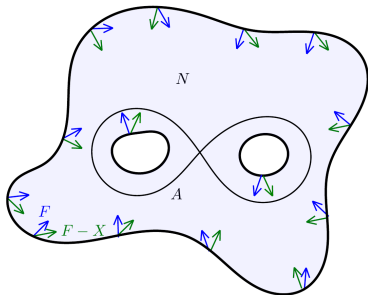
**Theorem (Poincaré, Hopf):** if  $N$  is a compact smooth manifold with boundary  $\partial N$ , then  $\chi(N) = 0 \iff$  there exists a nowhere-zero smooth vector field on  $N$  pointing inward at  $\partial N$ .

---

<sup>4</sup>Figures from Quanta Magazine.

## Generalization of Brockett's test

**Theorem (MDK & Koditschek 2022):** Let  $A \subset M$  be compact & stabilizable. Then  $\chi(A)$  is well-defined. If  $\chi(A) \neq 0$ , then for any sufficiently small vector field  $X$ ,  $X(x_0) = f(x_0, u_0)$  for some  $x_0, u_0$ .



**Proof:** Assume  $\exists$  stabilizing  $u(x)$  and define  $F(x) := f(x, u(x))$ . Lyapunov function theory  $\implies \exists$  compact smooth domain  $N \supset A$  s.t.  $F$  points inward at  $\partial N$  and  $\chi(A) = \chi(N) \neq 0$ . Continuity  $\implies F - X$  points inward at  $\partial N$  if  $X$  is small  $\implies F - X$  has a zero by Poincaré-Hopf  $\implies \exists x_0$  s.t.  $X(x_0) = F(x_0) = f(x_0, u(x_0))$ .

## Examples

### Heisenberg system

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv\end{aligned}\quad (2)$$

### Kinematic differential drive robot

$$\begin{aligned}\dot{x} &= u \cos \theta \\ \dot{y} &= u \sin \theta \\ \dot{\theta} &= v\end{aligned}\quad (3)$$

The right side of (2)  $\neq X_\varepsilon := (0, 0, \varepsilon)$  for any  $\varepsilon > 0$ .

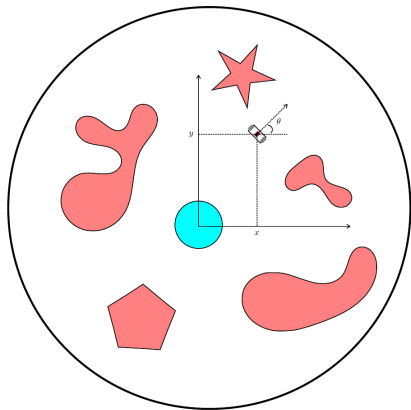
The right side of (3)  $\neq X_\varepsilon := (\varepsilon \sin \theta, -\varepsilon \cos \theta, 0)$  for any  $\varepsilon > 0$ .

Thus, our result  $\implies A$  is not stabilizable if  $\chi(A) \neq 0$ . E.g., if  $A$  is a stabilizable compact submanifold,  $A$  is a union of circles and tori.

**Other applications:** any stabilizable compact set has zero Euler characteristic for satellite orientation with  $\leq 2$  thrusters, for nonholonomic dynamics with  $\geq 1$  global constraint 1-form,...

## Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some “bad” set immediately enter some “good” set.



E.g., impossible for this differential drive robot to aim within  $\pm 179$  degrees of the origin while “strictly” avoiding obstacles via  $u(x)$ .

## Homotopy theorem & generalized Coron, Mansouri tests

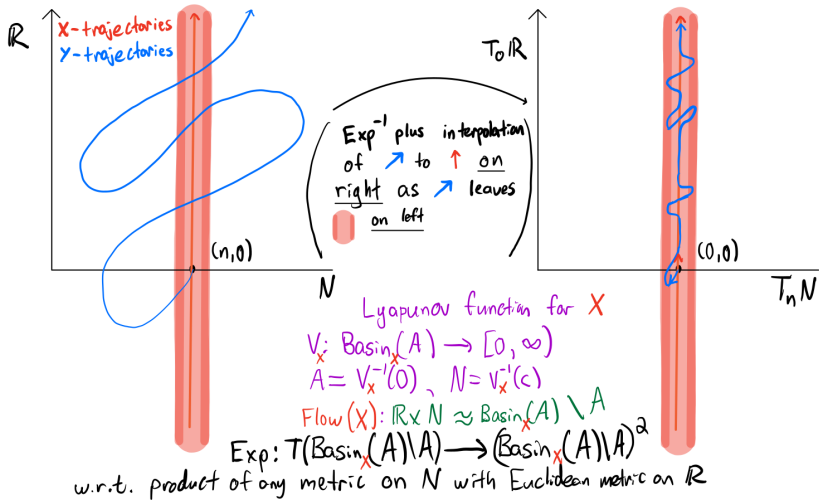
**Homotopy theorem (MDK 2023).** Let  $X, Y$  be smooth vector fields on a manifold  $M$  with a compact set  $A \subset M$  asymptotically stable for both. There is an open set  $U \supset A$  such that  $X|_{U \setminus A}, Y|_{U \setminus A}$  are homotopic through nowhere-zero vector fields.

$\implies$  **Theorem (MDK 2023).** Let the compact set  $A \subset M$  be asymptotically stable for *some* smooth vector field  $Y$  on  $M$ . If  $A$  is stabilizable for  $\dot{x} = f(x, u)$ , then for all small enough open  $U \supset A$ ,

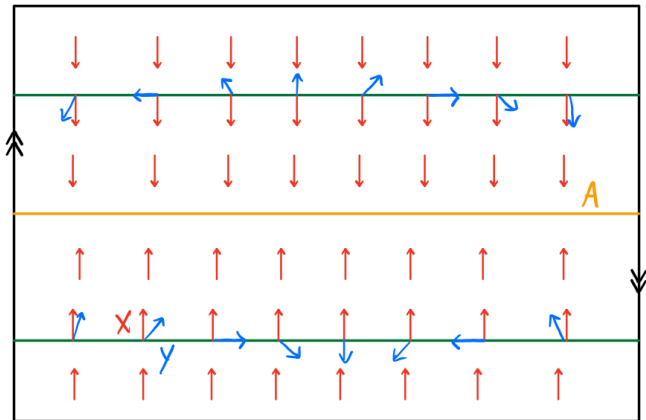
$$H_{\bullet}(T(U \setminus A) \setminus 0) \supset \underbrace{f_{*}H_{\bullet}(\Sigma) \supset Y_{*}H_{\bullet}(U \setminus A)}_{\text{cf. Coron, Mansouri}}.$$

These are stronger than all preceding results: there is an example (MDK 2023) for which non-stabilizability is detected by each of these theorems but not by any of the preceding theorems.

# Proof of the homotopy theorem



# Möbius strip example



$X \neq Y$  since  $Y \curvearrowright$  twice  
 around  $\bigcirc$  w.r.t.  $X$  while  $X \curvearrowright$   
 zero times w.r.t.  $X \Rightarrow$   $A$  is not  
asymptotically stable for  $Y$  by the  
 homotopy theorem.

## Can these results detect stabilizability of periodic orbits?

If  $A$  is the image of a periodic orbit with the same orientation for  $X$  and  $Y$ , the straight-line homotopy over a sufficiently small open  $U \supset A$  satisfies the homotopy theorem's conclusion regardless of whether  $A$  is attracting, repelling, or neither for  $X$  or  $Y$ .

**$\implies$  homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, preceding results also give no information.

...Could it be that periodic orbits might be “easy” to stabilize?



## Periodic orbits are sometimes easier to stabilize

Indeed, at least sometimes:

**Theorem (Anthony M. Bloch & MDK, in preparation).**

For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!

# Discovering engineering (im)possibilities for:

## Feedback stabilizability

- Brockett's necessary condition and beyond

- A homotopy theorem beyond the Coron/Mansouri tests

- Periodic orbits can be easier to stabilize than equilibria

## Applied Koopman operator methods

## Deep neural network autoencoders

# Discovering engineering (im)possibilities for:

Feedback stabilizability

Applied Koopman operator methods

Many assume the dynamical system is globally linearizable

Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

Deep neural network autoencoders

# “Modern Koopman theory for dynamical systems”

$$\dot{x} = \frac{d}{dt}x = f(x), \quad x \in M$$

- ▶ “A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.”<sup>5</sup>
- ▶ I.e., find an embedding  $F: M \hookrightarrow \mathbb{R}^n$  such that  $y = F(x)$  satisfies  $\dot{y} = By$  for some  $n \times n$  matrix  $B$ , or equivalently

$$\forall t \in \mathbb{R}: F \circ \Phi^t = e^{Bt} \circ F, \quad \text{where } \Phi = \text{flow}(f).$$

- ▶ Can then make linear predictions  $F(x(t)) = e^{Bt}F(x(0))$  of  $x(t)$  from  $x(0)$ , modulo nonlinearity from applying  $F$  and  $F^{-1}$ .
- ▶ **Fundamental question:** when is  $(M, \Phi)$  globally linearizable?

---

<sup>5</sup>Brunton, Budišić, Kaiser, and Kutz. SIAM Review, 64.2 (2022).

## When is a dynamical system $(M, \Phi)$ globally linearizable?<sup>7</sup>

**Not when  $M$  is connected and  $\Phi$  has a compact non-global attractor  $A$** , since  $\text{basin}(A)$  would then be closed (by the Jordan normal form theorem) in addition to open, hence clopen, so  $\text{basin}(A) = M$ .<sup>6</sup>

Hence we study global linearizability of the restriction  $(S, \Phi)$  of  $\Phi$  to a basin  $S$  of a compact attractor  $A$ ; we also study the important case that  $S$  is any compact invariant set for  $\Phi$ .

In these cases we obtain **necessary and sufficient conditions for global linearizability** of  $(S, \Phi)$  by an embedding, for the two cases of topological and smooth embeddings.

---

<sup>6</sup>This observation is complementary to Cor. 3 of Liu-Ozay-Sontag (2023).

<sup>7</sup>MDK and P. Arathoon, *Linearizability of flows by embeddings* (2023).

## Preliminaries

$F: S \rightarrow \mathbb{R}^n$  is a **topological embedding** if  $F$  is a one-to-one continuous map with a continuous inverse  $F^{-1}: F(S) \rightarrow S$ , and is a **smooth embedding** if additionally  $F$  and  $F^{-1}$  are smooth.

The  $n$ -torus  $T = T^n$  is Lie group isomorphic to  $(\mathbb{R}/\mathbb{Z})^n$ , vectors with  $n$  real entries but with addition defined elementwise modulo 1.

A **torus action** on  $S$  is a map  $\Theta: T \times S \rightarrow S$  satisfying  $\Theta^{\tau_1 + \tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$  for all  $s \in S$  and  $\tau_1, \tau_2 \in T$ .

The flow  $(S, \Phi)$  is a **1-parameter subgroup of a torus action** if  $\Phi^t = \Theta^{\omega t \bmod 1}$  for some torus action  $\Theta$  on  $S$ ,  $\omega \in \mathbb{R}^n \cong \text{Lie}(T^n)$ .

## The linearizability theorem, part 1: compact invariant sets

**Observation:** If  $(S, \Phi)$  is linearizable with  $S$  compact, Jordan normal form theorem implies  $(S, \Phi)$  embeds into flow on  $\mathbb{C}^n$  of a diagonal imaginary matrix, so  $(S, \Phi)$  is 1-parameter subgroup of restriction of standard torus action of  $T^n$  on  $\mathbb{C}^n$  to a subtorus.

This gives one implication below; Mostow-Palais gives the other.

**Theorem (MDK and P. Arathoon).** If  $S$  is a compact submanifold,  $(S, \Phi)$  is linearizable by a **smooth** embedding  $\iff (S, \Phi)$  is a 1-parameter subgroup of a **smooth** torus action.

**Theorem (MDK and PA).** If  $S$  is compact,  $(S, \Phi)$  is linearizable by a **topological** embedding  $\iff (S, \Phi)$  is a 1-parameter subgroup of a **continuous** torus action with **finitely many orbit types**<sup>8</sup>.

---

<sup>8</sup>I.e., there are only finitely many subgroups  $H \subset T$  such that  $H = \{\tau \in T : \Theta^\tau(s) = s\}$  is the set that fixes some  $s \in S$ .

## Implications concerning linearizability and topology

If  $(S, \Phi)$  is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an  $n \times n$  skew matrix  $B_e$  and a system of local coordinates on a neighborhood of each equilibrium  $e \in S$  such that  $\Phi^t \approx e^{B_e t}$ . Hence if  $e$  is isolated then  $B_e$  is invertible,  $n = \dim S$  is even, and the Hopf index of  $e$  is  $+1$ .

**Corollary (MDK and PA).** If  $S$  is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then  $(S, \Phi)$  cannot be linearized by a smooth embedding.

**Corollary (MDK and PA).**<sup>9</sup> If  $S$  is a compact submanifold containing at most finitely many equilibria such that  $(S, \Phi)$  is linearizable by a smooth embedding,  $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$ .

---

<sup>9</sup>Apply the Poincaré-Hopf theorem.



## Another point of view: quasiperiodic pinched torus families

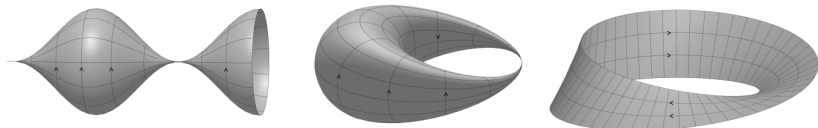


Figure: examples of quasiperiodic pinched torus families

**Definition.**  $P$  is a **pinched torus family** if there are  $m, n \in \mathbb{N}$ , closed subsets  $C_1, \dots, C_n \subset S \subset T^m$ , and a continuous group homomorphism  $F: T^n \rightarrow T^m$  such that  $P$  is the quotient of  $F^{-1}(S)$  by collapsing the  $j$ -th  $(\mathbb{R}/\mathbb{Z})$ -factor of  $F^{-1}(C_j) \subset T^n$  for all  $j$ . A pinched torus family  $P$  is **quasiperiodic** if it is equipped with the induced flow generated by any  $\omega \in \mathbb{R}^n$  with  $TF(\omega) = 0$ .

**Proposition (MDK and PA).** If  $S$  is compact,  $(S, \Phi)$  is linearizable by a **topological** embedding  $\iff (S, \Phi)$  is a **quasiperiodic pinched torus family**.

## The linearizability theorem, part 2: $S = \text{basin}(A)$

If  $S$  is the basin of an asymptotically stable compact set  $A \subset S$ ,  $A$  has continuous (smooth) **asymptotic phase**<sup>10</sup> if there is a continuous (smooth) **asymptotic phase map**  $P: S \rightarrow A$  satisfying

$$P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

**Theorem (MDK and PA).**  $(S, \Phi)$  is linearizable by a **topological** embedding  $\iff A$  has **continuous** asymptotic phase &  $(A, \Phi)$  is a 1-parameter subgroup of a **continuous** torus action with **finitely many orbit types**.

**Theorem (MDK and PA).**  $(S, \Phi)$  is linearizable by a **smooth** embedding  $\iff A$  is an **embedded submanifold** with **smooth** asymptotic phase,  $(A, \Phi)$  is a 1-parameter subgroup of a **smooth** torus action, & **locally  $\Phi \hookrightarrow$  reducible lin. flow on vector bundle...**

---

<sup>10</sup>This notion has roots in oscillator theory, and more generally NHIM theory.

## Questions remain

**Theorem (MDK and PA).**  $(S, \Phi)$  is linearizable by a smooth embedding  $\iff \dots$ , & for some open  $U \supset A$ ,  $(U, \Phi)$  embeds in a reducible linear flow covering  $\Phi$  on some vector bundle over  $A$ .

**When is this the case?** MDK and Revzen, Physica D (2021) give fairly complete answers<sup>11</sup> in the special cases that  $A$  is an equilibrium or periodic orbit, and some is known when  $A$  is a quasiperiodic torus, but this remains an open question in general.

**A necessary condition** for  $A$  to satisfy all conclusions of the theorem is that  $A$  be an (eventually relatively  $\infty$ -) **normally hyperbolic invariant manifold**. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

---

<sup>11</sup>involving “nonresonance” and “spectral spread” conditions on eigenvalues or Floquet multipliers of the infinitesimal linearization of the dynamics at  $A$

# Discovering engineering (im)possibilities for:

Feedback stabilizability

Applied Koopman operator methods

Many assume the dynamical system is globally linearizable

Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

Deep neural network autoencoders

# Discovering engineering (im)possibilities for:

Feedback stabilizability

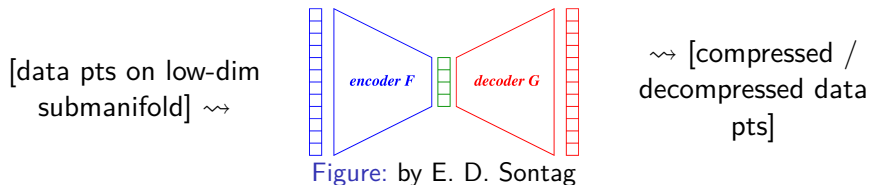
Applied Koopman operator methods

Deep neural network autoencoders

They can't work and yet they do: resolving the paradox

Just delete a thickened cut locus or top of a Morse complex

## (deep NN) Autoencoding and topological obstructions to it



- ▶ “Manifold hypothesis” postulates data set  $\subset \mathbb{R}^n$  lies on some  $k$ -dim submanifold  $K$ , describable locally by  $k < n$  parameters
- ▶ For  $K$  linear, classical approaches like PCA / MDS work well
- ▶  $K$  nonlinear, more challenging “manifold learning” problem
- ▶ Popular approach: look for **autoencoder**  $G \circ F$ , where the **encoder** output  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is the desired  $k$ -parameters,  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$  is the **decoder**, and  $F, G$  are continuous
- ▶ **Ideal autoencoders**:  $G(F(x)) = x$  for all  $x \in K$
- ▶ These **do not usually exist!** Since existence  $\implies k$ -dim  $K$  topologically embeds in  $\mathbb{R}^k$ , which is not true of most  $k$ -dim  $K$

# If autoencoding can't work, why does it?<sup>12</sup> Example:

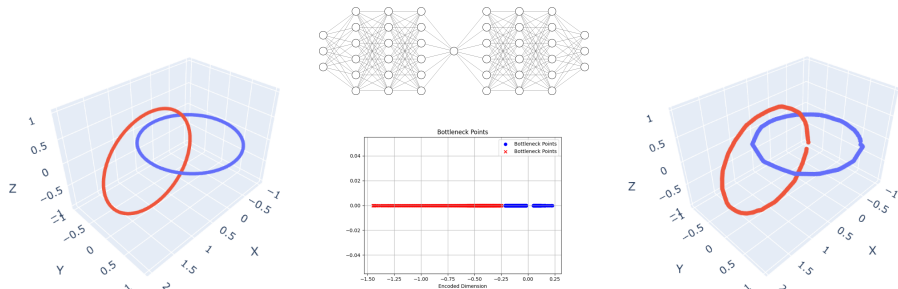


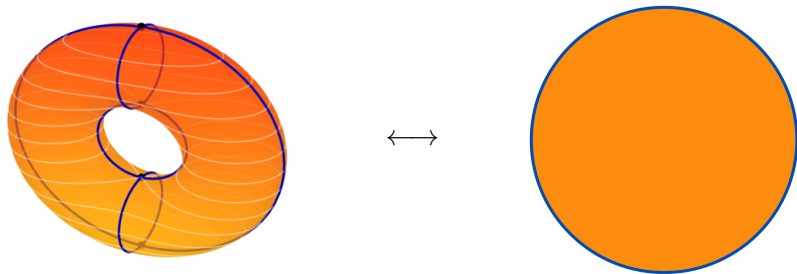
Figure: by E. D. Sontag

## Explanation:

- ▶ A pair of circles  $\subset \mathbb{R}^3$ , after thickening then deleting small intervals, is diffeomorphic to a pair of intervals  $\subset \mathbb{R}$
- ▶ **Encoder**  $F: \mathbb{R}^3 \rightarrow \mathbb{R}$  can be any extension of this diffeomorphism. **Decoder**  $G: \mathbb{R} \rightarrow \mathbb{R}^3$  can be any extension of the inverse diffeomorphism
- ▶ Can always find such small intervals disjoint from the data set

<sup>12</sup>MDK and E. D. Sontag, *Why do autoencoders work?* (2023).

If autoencoding can't work, how does it? More generally:



Explanation:

- ▶ A  $k$ -dim manifold  $\subset \mathbb{R}^n$ , after thickening then deleting either (i) a cut locus or (ii) the “top” cells from the complex of a polar Morse function<sup>13</sup>, is diffeomorphic to a  $k$ -dim disk  $\subset \mathbb{R}^k$
- ▶ **Encoder**  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  can be any extension of this diffeomorphism. **Decoder**  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$  can be any extension of the inverse diffeomorphism
- ▶ Can always find such a “disk boundary” disjoint from the data

---

<sup>13</sup>A *navigation function*, in the parlance of Rimon and Koditschek; these exist by Thm 3 in *Robot navigation functions on manifolds with boundary* (1990).



## Almost-ideal autoencoders always exist

$\mathcal{F}^{\ell,m} \subset \{\text{continuous funcs } \mathbb{R}^{\ell} \rightarrow \mathbb{R}^m\}$  are possible neural outputs.

**Theorem 1 (MDK and E. D. Sontag).** Let  $K \subset \mathbb{R}^n$  be a finite union of compact submanifolds with(out) boundary, each of dimension  $\leq k$ . For each  $\varepsilon, \delta > 0$  there is a closed set  $K_0 \subset K$  with intrinsic measure  $\mu(K_0) < \delta$  and  $F \in \mathcal{F}^{n,k}$ ,  $G \in \mathcal{F}^{k,n}$  such that length, surface area,...

$$\sup_{x \in K \setminus K_0} \|G(F(x)) - x\| < \varepsilon. \quad (4)$$

Moreover,  $K_0$  can be chosen disjoint from any finite set  $S \subset K$  and such that  $M \setminus K_0$  is connected for each component  $M$  of  $K$ .

**Q.** Can  $K_0$  be taken “smaller”, e.g., measure zero? **A.** No, by...

## Optimality of the almost-ideal autoencoding theorem

**Theorem 2 (MDK and EDS).** Let  $K \subset \mathbb{R}^n$  be a  $k$ -dimensional compact submanifold without boundary. For any continuous functions  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  and  $G: \mathbb{R}^k \rightarrow \mathbb{R}^n$ ,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\text{reach}} > 0. \quad (2)$$

**Proof:**

- ▶  $N_{r_K}(K) := \{x \in \mathbb{R}^n : \text{dist}(x, K) < r_K\}$  contains line segment from  $x \in N_{r_K}(K)$  to nearest  $\rho(x) \in K$ ;  $\rho$  is continuous.
- ▶ If (2) does not hold,  $t \mapsto \rho \circ (tG \circ F|_K + (1-t)\text{id}_K)$  is a homotopy of  $\text{id}_K$  to  $\rho \circ G \circ F|_K$ , so  $\text{deg}_{(2)}(\rho \circ G \circ F|_K) = 1$ .
- ▶  $\implies$  contradiction, since

$$0 = \text{deg}_{(2)}(\rho \circ G \circ F|_K) \sim \underbrace{(\rho \circ G \circ F|_K)^*}_{\parallel} : \check{H}^k(K) \rightarrow \check{H}^k(K)$$

$$\underbrace{(F|_K)^* \circ (G|_{F(K)})^* \circ \rho^*}_{0}$$

as  $\text{domain}[(G|_{F(K)})^*] = \check{H}^k(F(K)) = 0$  by duality theory.

# Discovering engineering (im)possibilities for:

Feedback stabilizability

Applied Koopman operator methods

Deep neural network autoencoders

They can't work and yet they do: resolving the paradox

Just delete a thickened cut locus or top of a Morse complex

Thank you for your time and attention.

# Discovering engineering (im)possibilities for:

## Feedback stabilizability

Brockett's necessary condition and beyond

A homotopy theorem beyond the Coron/Mansouri tests

Periodic orbits can be easier to stabilize than equilibria

## Applied Koopman operator methods

Many assume the dynamical system is globally linearizable

Which ones are?

1-parameter subgroups of torus actions with asymptotic phase

## Deep neural network autoencoders

They can't work and yet they do: resolving the paradox

Just delete a thickened cut locus or top of a Morse complex