On the universality of linear dynamics[1]

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Ingredients for the universality problem

A **flow** on a set X is a map $\Phi\colon \mathbb{R} \times X \to X$ satisfying

- $\Phi^0 = \mathrm{id}_X$, where $\Phi^t := \Phi(t, \cdot) : X \to X$, and
- $\Phi^t \circ \Phi^s = \Phi^{t+s}$.

Fix $k \in \mathbb{N}_{\geq 0} \cup \{\infty\}$. Let X be a topological space if $k = 0$ and a C^k manifold if $k \geq 1$.

A map $h\colon X \to Y$ is a C^0 **embedding** if $h\colon X \to h(X)$ is a homeomorphism, and a $C^{k \geq 1}$ embedding if $h(X)$ is a C^k manifold and $h\colon X \to h(X)$ is a C^k **diffeomorphism**.

Universality problem in general[2]

A class of flows $\mathcal{C}$ is C^k-**universal** for another such class $\mathcal{D}$ if any flow (X, Φ) in $\mathcal{D}$ admits an equivariant C^k embedding into some flow (Y, Ψ) in $\mathcal{C}$, meaning there is a C^k embedding $h\colon X \hookrightarrow Y$ s.t.

$$\forall t \in \mathbb{R}\colon h \circ \Phi^t = \Psi^t \circ h.$$

Conceptually, such an embedding C^k-identifies (X, Φ) with the Ψ-invariant subset $h(X) \subset Y$ (submanifold for $k \geq 1$).

C^k **universality problem.** Given $\mathcal{C}$, what is the largest class of flows $\mathcal{D}$ such that $\mathcal{C}$ is C^k-universal for $\mathcal{D}$?

Example. T Tao [1] solved the C^∞ universality problem for

$$\mathcal{C} = \{\text{potential well systems } \ddot{x} = -\nabla V(x)\}.$$

[2] This slightly generalizes definitions of T Tao [1], R Cardona & F Presas [2], and Á González-Prieto, Miranda, & Peralta-Salas [3].

Linear universality problem

From here on, this talk concerns the C^k universality problem for

$$\mathcal{C} = \{\text{linear dynamical systems } \dot{x} = Ax \text{ on Euclidean spaces}\}.$$

This **linear universality problem** is, equivalently, the C^k **global linearizability problem**:

"What class $\mathcal{D}$ of flows are globally linearizable by C^k embeddings?"

Why care?

- Linear dynamical systems are arguably the "simplest" ones.
- "Applied Koopman"[3] algorithms like "Extended Dynamic Mode Decomposition" implicitly *assume* global linearizability.
- Equivalently, the question is (G D Mostow [5], R S Palais [6]): "To what extent does the Mostow-Palais equivariant embedding theorem extend to group actions of $\mathbb{R}$?"

[3] See, e.g., S L Brunton et al. [4] for a relatively recent survey.

Some sufficient conditions for global linearizability[4]

A **locally** linearizing C^0 embedding always exists near a hyperbolic equilibrium or periodic orbit of a C^1 flow (Hartman-Grobman, Floquet).

A **globally** linearizing C^0 embedding always exists:

- if (X, Φ) has a globally *exponentially* stable equilibrium or periodic orbit (Y Lan & I Mezić [7], MDK & S Revzen [8]);
- if (X, Φ) has a globally *asymptotically* stable equilibrium (MDK & E D Sontag [9])—*not necessarily hyperbolic*.

Note. In non-globally attracting cases, these results still apply to restrictions of flows to basins of attraction.

Note. The embeddings are dimension-*increasing* for periodic orbits, but dimension-*preserving* for equilibria (homeomorphisms).

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[4] There are C^k versions of these results under further assumptions (nonresonance & spectral spread, C^k dynamics).

Dimension-increasing embeddings $\implies$ new possibilities

Various classical impossibilities, such as existence of linearizing homeomorphisms in the presence of multiple isolated equilibria, become possible for dimension-*increasing* linearizing embeddings.

Theorem (P Arathoon & MDK [10]). For any $n > 1$ there is a C^∞ flow $(\mathbb{R}^n, \Phi)$ with any given finite number of isolated equilibria that is linearizable by a C^∞ embedding.

In fact, the embedding can be taken of the form $h(x) = (x, g(x))$![5]

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[5]Linearizing embeddings of this special form were studied by D Claude, M Fliess, and A Isidori [11] and recently M-A Belabbas and X Chen [12].

Example[6]

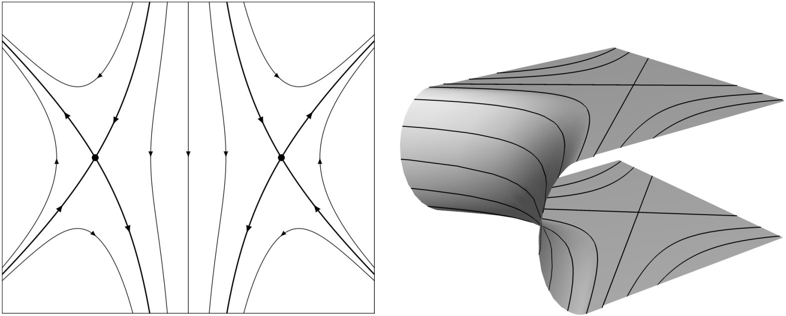

Figure: C^∞ embedding of a nonlinear system on $\mathbb{R}^2$ with two isolated equilibria into a linear system on $\mathbb{R}^3$.

[6] P Arathoon & MDK [10].

Some necessary conditions for global linearizability[7]

On the other hand, a linearizing C^0 embedding cannot exist

- if there is a countable number > 1 of omega-limit sets—such as isolated equilibria—*and all forward trajectories are precompact* (Z Liu, N Ozay, & E D Sontag [13, 14]), or
- if there is more than one nonempty compact attractor or, more generally, a single nonempty non-global compact attractor (MDK & P Arathoon [15]).

[7] In fact, these results also hold for linearizability by merely injective C^0 maps if X is homeomorphic to a subspace of a manifold.

Preamble to necessary and sufficient conditions

Assume all state spaces are connected to avoid trivialities.

- We can provide necessary and sufficient conditions for linearizability of (X, Φ) by a C^k embedding if there is at least one nonempty compact attractor.
- In other words, we can solve the C^k universality problem for

  $$\mathcal{C} = \{\text{linear dynamical systems } \dot{x} = Ax \text{ on Euclidean spaces}\}$$

  under the constraint that $\mathcal{D} \subset \mathcal{E}$, where $\mathcal{E}$ consists of flows having at least one nonempty compact attractor.
- Since (X, Φ) is not linearizable by a C^0 embedding if there is a nonempty non-global compact attractor (MDK & P Arathoon [15]), it remains to consider the case of a global compact attractor.[8]

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[8] This includes the case of a compact state space. Also, the results can be applied after restriction to the basin of a local attractor.

Torus preliminaries

An m-**torus** $T = T^m$ is Lie group isomorphic to $(\mathbb{R}/\mathbb{Z})^m$, vectors with m real entries but with addition defined elementwise mod 1.

A **torus action** on a space S is a map $\Theta\colon T \times S \to S$ satisfying $\Theta^0 = \mathrm{id}_S$ & $\Theta^{\tau_1+\tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

A **1-parameter subgroup** of Θ is a map $\Phi\colon \mathbb{R} \times S \to S$ of the form $\Phi^t(x) = \Theta^{\omega t}(x)$ for some $\omega \in \mathbb{R}^m$.

Θ has **finite orbit types** if there are only finitely many subgroups $H \subset T$ such that, for some $x \in S$,

$$H = \mathrm{Stabilizer}(x) := \{\tau \in T\colon \Theta^\tau(x) = x\}.$$

First, special case: compact state spaces[9]

Theorem. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Let Φ be a C^k flow on a compact C^k manifold X. Then (X, Φ) is linearizable by a C^k embedding $\iff$ Φ is a 1-parameter subgroup of a C^k torus action.

Theorem. Let X be a compact topological space homeomorphic to a subspace of a manifold, and Φ be a C^0 flow on X. Then (X, Φ) is linearizable by a C^0 embedding $\iff$ Φ is a 1-parameter subgroup of a C^0 torus action with finite orbit types.

Note. For noncompact X, the theorems be applied by restricting to compact invariant sets.

[9] MDK & P Arathoon [15].

Some $C^{k\geq 1}$ corollaries[10]

Preceding $C^{k\geq 1}$ theorem + Bochner's linearization theorem

$$\Downarrow$$

Proposition. Assume that (X, Φ) can be linearized by a C^1 embedding and Φ has at least one isolated equilibrium. Then X is even-dimensional, and the Hopf index of the generating vector field at any isolated equilibrium is equal to $+1$.

$$\Downarrow$$

Corollary. Assume that X is odd-dimensional and Φ has at least one isolated equilibrium. Then (X, Φ) is not linearizable by a C^1 embedding.

Corollary Assume that Φ has only finitely many equilibria and that (X, Φ) is linearizable by a C^1 embedding. Then the Euler characteristic $\chi(X) = \#\{\text{equilibria}\} \geq 0$.

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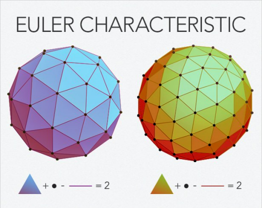
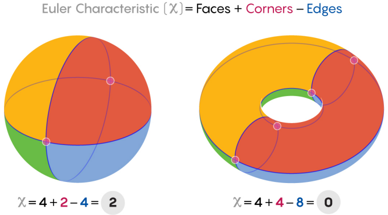
[10] MDK & P Arathoon [15].

An Euler characteristic primer[11]

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).





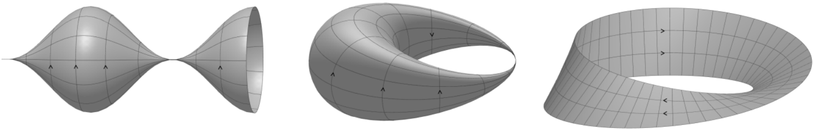
Notation: $\chi(Y) :=$ Euler characteristic of Y.

Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\Sigma_g) = 2 - 2g$

Σ_g for $g = 1, 2, 3$ (For $g > 1$, flows on these are not linearizable by C^1 embeddings if all equilibria are isolated).

[11] Figures from Quanta Magazine and Wikipedia.

Some C^0 examples and a C^∞ example[12]

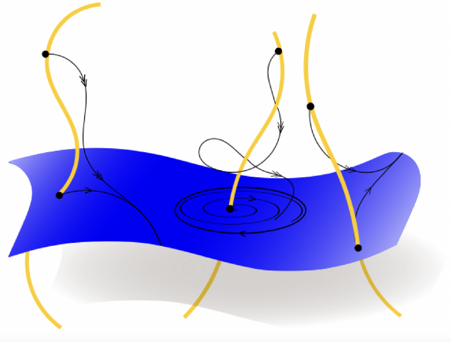


Preceding C^0 theorem implies that the first two flows are linearizable by C^0 embeddings. Preceding $C^{k\geq 1}$ theorem implies that the third flow is linearizable by a C^∞ embedding.

This is easy to see since all flows are 1-parameter subgroups of torus (circle) actions.

[12] MDK and P Arathoon [15].

Preamble to the general case: asymptotic phase



If $A \subset X$ is a global compact attractor for (X, Φ), an **asymptotic phase map** is a retraction $P \colon X \to A$ ($P|_A = \mathrm{id}_A$) such that

$$\forall t \in \mathbb{R} \colon P \circ \Phi^t = \Phi^t \circ P$$

If P is C^0, it follows readily that

$$\mathrm{dist}(\Phi^t(x), \Phi^t(P(x))) \to 0 \quad \text{as} \quad t \to \infty.$$

Thus, x is "**asymptotically in phase with**" $P(x)$.

General C^0 global attractor case[13]

Theorem. Let X be a locally compact topological space homeomorphic to a subspace of a manifold. Let Φ be a C^0 flow on X with a globally asymptotically stable compact invariant set A. Then (X, Φ) is linearizable by a C^0 embedding if and only if:

1. there is a C^0 asymptotic phase map $P\colon X \to A$, and
2. $\Phi|_{\mathbb{R}\times A}$ is a 1-parameter subgroup of a C^0 torus action with finite orbit types.

Moreover, any linearizing injective C^0 map is proper, hence also a C^0 embedding.

Note. For locally asymptotically stable A, the theorem can be applied by restricting to the basin of attraction of A.

[13] MDK & P Arathoon [15].

Application of preceding theorem to periodic orbits

Corollary. The flow on the basin of an asymptotically stable periodic orbit is linearizable by a C^0 embedding $\iff$ there is C^0 asymptotic phase.

Example. Using polar coordinates (r, θ) on $\mathbb{R}^2$, the system

$$\dot{r} = -(r-1)^3, \qquad \dot{\theta} = r$$

generates a smooth flow Φ on $\mathbb{R}^2 \setminus \{0\}$ with globally asymptotically stable limit cycle $A = \{r = 1\}$. Closed-form expression for $\Phi \implies$

$$\operatorname{dist}(\Phi^t(x), \Phi^t(y)) \not\to 0 \quad \text{as} \quad t \to \infty$$

for any $x \notin A$, $y \in A \implies A$ does not have C^0 asymptotic phase $\implies$ a linearizing C^0 embedding **does not exist**.[14]

[14] By the final sentence of the preceding theorem, even a linearizing injective C^0 map does not exist.

General $C^{k\geq 1}$ global attractor case[15]

Theorem. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Let Φ be a C^k flow of a uniquely integrable vector field on a C^k manifold X with a globally asymptotically stable compact invariant set $A \subset X$. Then (X, Φ) is linearizable by a C^k embedding if and only if:

1. A is a C^k embedded submanifold of X and there is a C^k asymptotic phase map $P : X \to A$;
2. the restricted flow $\Phi|_{\mathbb{R}\times A}$ is a 1-parameter subgroup of a C^k torus action; and
3. there is a C^k map $G\colon U \to \mathbb{R}^\ell$, a matrix $B \in \mathbb{R}^{\ell\times\ell}$ with all eigenvalues having negative real part, and an open set $U \supset A$ such that $\ker(\mathsf{T}_A G) = \mathsf{T}A$ and $G(\Phi^t(x)) = e^{Bt}G(x)$ for all $x \in U$ and $t \in \mathbb{R}$ such that $\Phi^t(x) \in U$.

Here $T_A X := TX|_A$ is the tangent bundle over A and $T_A G\colon T_A X \to T\mathbb{R}^\ell$ is the restriction $TG|_{T_A X}$ of the tangent map.

[15] MDK & P Arathoon [15].

Relationship to normal hyperbolicity

If (X, Φ) is linearizable by a $C^{k\geq 1}$ embedding, then A must in fact be a normally hyperbolic invariant manifold (NHIM). More precisely, A is an eventually relatively ∞-NHIM with respect to any Riemannian metric on X.[16]

This follows from the proof of the preceding theorem, which shows that (X, Φ) is linearizable by a proper C^k embedding $F\colon X \hookrightarrow \mathbb{R}^n$ with $F(A)$ contained in the real invariant subspace for the generator of the linear flow corresponding to eigenspaces of purely imaginary eigenvalues. The same reasoning implies that A satisfies "center bunching" conditions of all orders, consistent with the existence of C^k asymptotic phase implied by these conditions.[17]

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[16] See M W Hirsch, C C Pugh, & M Shub [16] and J Eldering, MDK, and S Revzen [17].

[17] See N Fenichel [18] and C C Pugh, M Shub, & A Wilkinson [19].

Thank you for your attention.

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General and linear universality problems

Sufficient conditions for global linearizability

Necessary conditions for global linearizability

Necessary and sufficient conditions for global linearizability