

On the universality of linear dynamics¹

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Ingredients for the universality problem

A **flow** on a set X is a map $\Phi: \mathbb{R} \times X \rightarrow X$ satisfying

- ▶ $\Phi^0 = \text{id}_X$, where $\Phi^t := \Phi(t, \cdot) : X \rightarrow X$, and
- ▶ $\Phi^t \circ \Phi^s = \Phi^{t+s}$.

Fix $k \in \mathbb{N}_{\geq 0} \cup \{\infty\}$. Let X be a topological space if $k = 0$ and a C^k manifold if $k \geq 1$.

A map $h: X \rightarrow Y$ is a C^0 **embedding** if $h: X \rightarrow h(X)$ is a homeomorphism, and a $C^{k \geq 1}$ embedding if $h(X)$ is a C^k manifold and $h: X \rightarrow h(X)$ is a C^k **diffeomorphism**.

Universality problem in general²

A class of flows $\mathcal{C}$ is C^k -**universal** for another such class $\mathcal{D}$ if any flow (X, Φ) in $\mathcal{D}$ admits an equivariant C^k embedding into some flow (Y, Ψ) in $\mathcal{C}$, meaning there is a C^k embedding $h: X \hookrightarrow Y$ s.t.

$$\forall t \in \mathbb{R}: h \circ \Phi^t = \Psi^t \circ h.$$

Conceptually, such an embedding C^k -identifies (X, Φ) with the Ψ -invariant subset $h(X) \subset Y$ (submanifold for $k \geq 1$).

C^k universality problem. Given $\mathcal{C}$, what is the largest class of flows $\mathcal{D}$ such that $\mathcal{C}$ is C^k -universal for $\mathcal{D}$?

Example. Tao [1] solved the C^∞ universality problem for

$$\mathcal{C} = \{\text{potential well systems } \ddot{x} = -\nabla V(x)\}.$$

²This slightly generalizes definitions of Tao [1], Cardona & Presas [2], and González-Prieto, Miranda, & Peralta-Salas [3].

Linear universality problem

From here on, this talk concerns the C^k universality problem for

$$\mathcal{C} = \{\text{linear dynamical systems } \dot{x} = Ax \text{ on Euclidean spaces}\}.$$

This **linear universality problem** is, equivalently, the C^k **global linearizability problem**:

“What class $\mathcal{D}$ of flows are globally linearizable by C^k embeddings?”

Why care?

- ▶ Linear dynamical systems are arguably the “simplest” ones.
- ▶ “Applied Koopman”³ algorithms like “Extended Dynamic Mode Decomposition” implicitly *assume* global linearizability.
- ▶ Equivalently, the question is (Mostow [5], Palais [6]):
“To what extent does the Mostow-Palais equivariant embedding theorem extend to group actions of $\mathbb{R}$?”

³See, e.g., Brunton et al. [4] for a relatively recent survey.

Some sufficient conditions for global linearizability⁴

A **locally** linearizing C^0 embedding always exists near a hyperbolic equilibrium or periodic orbit of a C^1 flow (Hartman-Grobman, Floquet).

A **globally** linearizing C^0 embedding always exists:

- ▶ if (X, Φ) has a globally *exponentially* stable equilibrium or periodic orbit (Lan & Mezić [7], K & Revzen [8]);
- ▶ if (X, Φ) has a globally *asymptotically* stable equilibrium (K & Sontag [9])—*not necessarily hyperbolic*.

Note. In non-globally attracting cases, these results still apply to restrictions of flows to basins of attraction.

Note. The embeddings are dimension-*increasing* for periodic orbits, but dimension-*preserving* for equilibria (homeomorphisms).

⁴There are C^k versions of these results under further assumptions (nonresonance & spectral spread, C^k dynamics).

Dimension-increasing embeddings $\implies$ new possibilities

Various classical impossibilities, such as existence of linearizing homeomorphisms in the presence of multiple isolated equilibria, become possible for dimension-*increasing* linearizing embeddings.

Theorem (Arathoon & K [10]). For any $n > 1$ there is a C^∞ flow $(\mathbb{R}^n, \Phi)$ with any given finite number of isolated equilibria that is linearizable by a C^∞ embedding.

In fact, the embedding can be taken of the form $h(x) = (x, g(x))$!⁵

⁵Linearizing embeddings of this special form were studied by Claude, Fliess, & Isidori [11] and recently Belabbas & Chen [12].

Example⁶

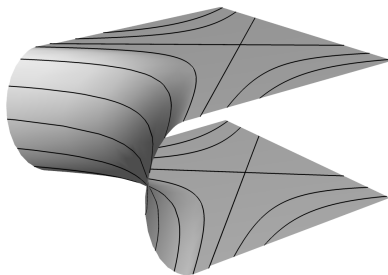
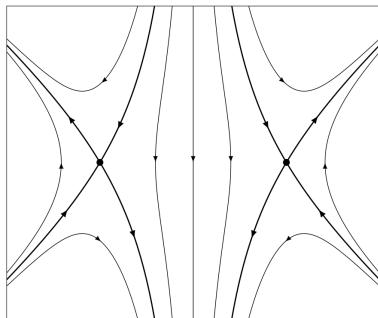


Figure: C^∞ embedding of a nonlinear system on $\mathbb{R}^2$ with two isolated equilibria into a linear system on $\mathbb{R}^3$.

⁶Arathoon & K [10].

Some necessary conditions for global linearizability⁷

On the other hand, a linearizing C^0 embedding cannot exist

- ▶ if there is a countable number > 1 of omega-limit sets—such as isolated equilibria—*and all forward trajectories are precompact* (Liu, Ozay, & Sontag [13, 14]), or
- ▶ if there is more than one nonempty compact attractor or, more generally, a single nonempty non-global compact attractor (K & Arathoon [15]).

⁷In fact, these results also hold for linearizability by merely injective C^0 maps if X is homeomorphic to a subspace of a manifold.

Preamble to necessary and sufficient conditions

Assume all state spaces are connected to avoid trivialities.

- ▶ We can provide necessary and sufficient conditions for linearizability of (X, Φ) by a C^k embedding if there is at least one nonempty compact attractor.

- ▶ In other words, we can solve the C^k universality problem for

$$\mathcal{C} = \{\text{linear dynamical systems } \dot{x} = Ax \text{ on Euclidean spaces}\}$$

under the constraint that $\mathcal{D} \subset \mathcal{E}$, where $\mathcal{E}$ consists of flows having at least one nonempty compact attractor.

- ▶ Since (X, Φ) is not linearizable by a C^0 embedding if there is a nonempty non-global compact attractor (K & Arathoon [15]), it remains to consider the case of a global compact attractor.⁸

⁸This includes the case of a compact state space. Also, the results can be applied after restriction to the basin of a local attractor.

Torus preliminaries

An **m -torus** $T = T^m$ is Lie group isomorphic to $(\mathbb{R}/\mathbb{Z})^m$, vectors with m real entries but with addition defined elementwise mod 1.

A **torus action** on a space S is a map $\Theta: T \times S \rightarrow S$ satisfying $\Theta^0 = \text{id}_S$ & $\Theta^{\tau_1 + \tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

A **1-parameter subgroup** of Θ is a map $\Phi: \mathbb{R} \times S \rightarrow S$ of the form $\Phi^t(x) = \Theta^{\omega t}(x)$ for some $\omega \in \mathbb{R}^m$.

Θ has **finite orbit types** if there are only finitely many subgroups $H \subset T$ such that, for some $x \in S$,

$$H = \text{Stabilizer}(x) := \{\tau \in T : \Theta^\tau(x) = x\}.$$

First, special case: compact state spaces⁹

Theorem. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Let Φ be a C^k flow on a compact C^k manifold X . Then (X, Φ) is linearizable by a C^k embedding $\iff \Phi$ is a 1-parameter subgroup of a C^k torus action.

Theorem. Let X be a compact topological space homeomorphic to a subspace of a manifold, and Φ be a C^0 flow on X . Then (X, Φ) is linearizable by a C^0 embedding $\iff \Phi$ is a 1-parameter subgroup of a C^0 torus action with finite orbit types.

Note. For noncompact X , the theorems be applied by restricting to compact invariant sets.

⁹K & Arathoon [15].

Some $C^{k \geq 1}$ corollaries¹⁰

Preceding $C^{k \geq 1}$ theorem + Bochner's linearization theorem



Proposition. Assume that (X, Φ) can be linearized by a C^1 embedding and Φ has at least one isolated equilibrium. Then X is even-dimensional, and the Hopf index of the generating vector field at any isolated equilibrium is equal to $+1$.



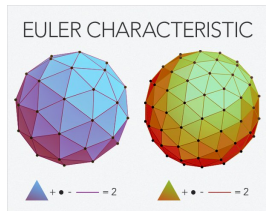
Corollary. Assume that X is odd-dimensional and Φ has at least one isolated equilibrium. Then (X, Φ) is not linearizable by a C^1 embedding.

Corollary Assume that Φ has only finitely many equilibria and that (X, Φ) is linearizable by a C^1 embedding. Then the Euler characteristic $\chi(X) = \#\{\text{equilibria}\} \geq 0$.

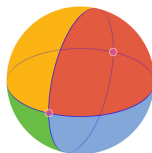
¹⁰K & Arathoon [15].

An Euler characteristic primer¹¹

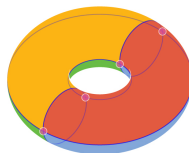
Goes back to Maurolico (1537), Euler (1758).



Euler Characteristic (χ) = Faces + Corners - Edges



$$\chi = 4 + 2 - 4 = 2$$



$$\chi = 4 + 4 - 8 = 0$$

Notation: $\chi(Y) :=$ Euler characteristic of Y .

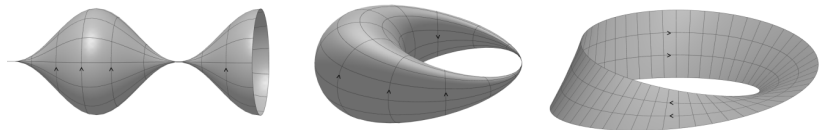
Examples: $\chi(\bullet) = 1$, $\chi(S^1) = 0$, $\chi(S^2) = 2$, $\chi(\Sigma_g) = 2 - 2g$



Σ_g for $g = 1, 2, 3$ (For $g > 1$, flows on these are not linearizable by C^1 embeddings if all equilibria are isolated).

¹¹Figures from Quanta Magazine and Wikipedia.

Some C^0 examples and a C^∞ example¹²

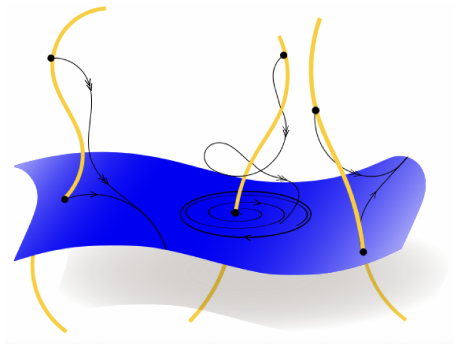


Preceding C^0 theorem implies that the first two flows are linearizable by C^0 embeddings. Preceding $C^{k \geq 1}$ theorem implies that the third flow is linearizable by a C^∞ embedding.

This is easy to see since all flows are 1-parameter subgroups of torus (circle) actions.

¹²K and Arathoon [15].

Preamble to the general case: asymptotic phase



If $A \subset X$ is a global compact attractor for (X, Φ) , an **asymptotic phase map** is a retraction $P: X \rightarrow A$ ($P|_A = \text{id}_A$) such that

$$\forall t \in \mathbb{R}: P \circ \Phi^t = \Phi^t \circ P$$

If P is C^0 , it follows readily that

$$\text{dist}(\Phi^t(x), \Phi^t(P(x))) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

Thus, x is “**asymptotically in phase with**” $P(x)$.

General C^0 global attractor case¹³

Theorem. Let X be a locally compact topological space homeomorphic to a subspace of a manifold. Let Φ be a C^0 flow on X with a globally asymptotically stable compact invariant set A . Then (X, Φ) is linearizable by a C^0 embedding if and only if:

1. there is a C^0 asymptotic phase map $P: X \rightarrow A$, and
2. $\Phi|_{\mathbb{R} \times A}$ is a 1-parameter subgroup of a C^0 torus action with finite orbit types.

Moreover, any linearizing injective C^0 map is proper, hence also a C^0 embedding.

Note. For locally asymptotically stable A , the theorem can be applied by restricting to the basin of attraction of A .

¹³K & Arathoon [15].

Application of preceding theorem to periodic orbits

Corollary. The flow on the basin of an asymptotically stable periodic orbit is linearizable by a C^0 embedding
 $\iff$ there is C^0 asymptotic phase.

Example. Using polar coordinates (r, θ) on $\mathbb{R}^2$, the system

$$\dot{r} = -(r - 1)^3, \quad \dot{\theta} = r$$

generates a smooth flow Φ on $\mathbb{R}^2 \setminus \{0\}$ with globally asymptotically stable limit cycle $A = \{r = 1\}$. Closed-form expression for $\Phi \implies$

$$\text{dist}(\Phi^t(x), \Phi^t(y)) \not\rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

for any $x \notin A, y \in A \implies A$ does not have C^0 asymptotic phase
 $\implies$ a linearizing C^0 embedding **does not exist**.¹⁴

¹⁴By the final sentence of the preceding theorem, even a linearizing injective C^0 map does not exist.

General $C^{k \geq 1}$ global attractor case¹⁵

Theorem. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Let Φ be a C^k flow of a uniquely integrable vector field on a C^k manifold X with a globally asymptotically stable compact invariant set $A \subset X$. Then (X, Φ) is linearizable by a C^k embedding if and only if:

1. A is a C^k embedded submanifold of X and there is a C^k asymptotic phase map $P : X \rightarrow A$;
2. the restricted flow $\Phi|_{\mathbb{R} \times A}$ is a 1-parameter subgroup of a C^k torus action; and
3. there is a C^k map $G : U \rightarrow \mathbb{R}^\ell$, a matrix $B \in \mathbb{R}^{\ell \times \ell}$ with all eigenvalues having negative real part, and an open set $U \supset A$ such that $\ker(T_A G) = TA$ and $G(\Phi^t(x)) = e^{Bt}G(x)$ for all $x \in U$ and $t \in \mathbb{R}$ such that $\Phi^t(x) \in U$.

Here $T_A X := TX|_A$ is the tangent bundle over A and $T_A G : T_A X \rightarrow T\mathbb{R}^\ell$ is the restriction $TG|_{T_A X}$ of the tangent map.

¹⁵K & Arathoon [15].

Relationship to normal hyperbolicity

If (X, Φ) is linearizable by a $C^{k \geq 1}$ embedding, then A must in fact be a normally hyperbolic invariant manifold (NHIM). More precisely, A is an eventually relatively ∞ -NHIM with respect to any Riemannian metric on X .¹⁶

This follows from the proof of the preceding theorem, which shows that (X, Φ) is linearizable by a proper C^k embedding $F: X \hookrightarrow \mathbb{R}^n$ with $F(A)$ contained in the real invariant subspace for the generator of the linear flow corresponding to eigenspaces of purely imaginary eigenvalues. The same reasoning implies that A satisfies “center bunching” conditions of all orders, consistent with the existence of C^k asymptotic phase implied by these conditions.¹⁷

¹⁶See Hirsch, Pugh, & Shub [16] and Eldering, K, & Revzen [17].

¹⁷See Fenichel [18] and Pugh, Shub, & Wilkinson [19].

Thank you for your attention.

References, part 1

- [1] T Tao. On the universality of potential well dynamics. *Dynamics of Partial Differential Equations*, 2017.
- [2] R Cardona and F Presas. An h-principle for embeddings transverse to a contact structure. *J. Topology*, 2024.
- [3] Á González-Prieto, E Miranda, and D Peralta-Salas. Universality in computable dynamical systems: old and new. *Journal of Physics: Complexity*, 2025.
- [4] S L Brunton, M Budišić, E Kaiser, and J N Kutz. Modern Koopman theory for dynamical systems. *SIAM Review*, 2022.
- [5] G D Mostow. Equivariant embeddings in Euclidean space. *Annals of Mathematics*, 1957.
- [6] R S Palais. Imbedding of compact, differentiable transformation groups in orthogonal representations. *Journal of Mathematics and Mechanics*, 1957.
- [7] Y Lan and I Mezić. Linearization in the large of nonlinear systems and Koopman operator spectrum. *Physica D*, 2013.

References, part 2

- [8] M D Kvalheim and S Revzen. Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits. *Physica D*, 2021.
- [9] M D Kvalheim and E D Sontag. Global linearization of asymptotically stable systems without hyperbolicity. *Systems & Control Letters*, 2025.
- [10] P Arathoon and M D Kvalheim. Koopman embedding and super-linearization counterexamples with isolated equilibria. *arXiv:2306.15126*, 2023.
- [11] D Claude, M Fliess, and A Isidori. Immersion, directe et par bouclage, d'un système non linéaire dans un linéaire. *C. R. Acad. Sci Paris Sér. I Math.*, 1983.
- [12] M-A Belabbas and X Chen. A sufficient condition for the super-linearization of polynomial systems. *Systems & Control Letters*, 2023.

References, part 3

- [13] Z Liu, N Ozay, and E D Sontag. On the non-existence of immersions for systems with multiple omega-limit sets. *IFAC-PapersOnLine*, 2023.
- [14] Z Liu, N Ozay, and E D Sontag. Properties of immersions for systems with multiple limit sets with implications to learning Koopman embeddings. *Automatica*, 2025.
- [15] M D Kvalheim and P. Arathoon. Linearizability of flows by embeddings. *Selecta Mathematica*, 2026. (arXiv:2305.18288.)
- [16] M W Hirsch, C C Pugh, and M Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Springer-Verlag, 1977.
- [17] J Eldering, M D Kvalheim, and S Revzen. Global linearization and fiber bundle structure of invariant manifolds. *Nonlinearity*, 2018.
- [18] N Fenichel. Asymptotic stability with rate conditions, II. *Indiana University Mathematics Journal*, 1977.
- [19] C C Pugh, M Shub, & A Wilkinson. Hölder foliations. *Duke Mathematics Journal*, 1997.

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General and linear universality problems

Sufficient conditions for global linearizability

Necessary conditions for global linearizability

Necessary and sufficient conditions for global linearizability