Identifying engineering (im)possibilities with geometry and topology[1]

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Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

Feedback stabilizability

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Deep neural network autoencoders

- They should not work, and yet they do: resolving the paradox
- Training implications: L^2 but not L^∞ error can be made small

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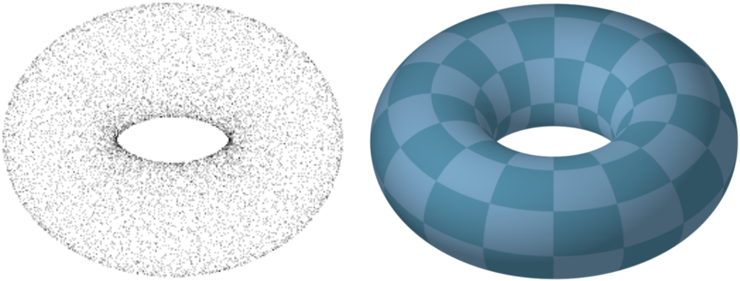
Dimensionality reduction of data

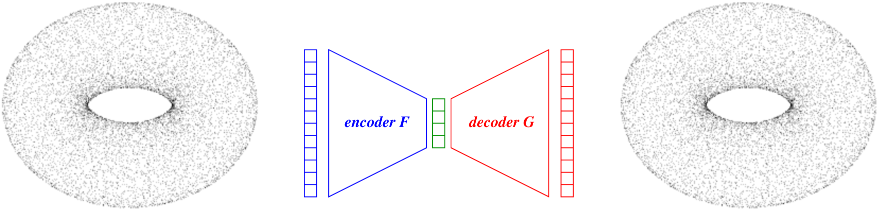


The "manifold hypothesis" postulates that a data set in $\mathbb{R}^n$ lies on some k-dimensional submanifold $K \subset \mathbb{R}^n$.

$\implies$ data can be parametrized locally by $k < n$ real numbers.

Classical approaches like PCA to learn these parameters work well when K is linear, but not when K is nonlinear.

Autoencoding as a nonlinear dimensionality reduction approach (and why it should not work)



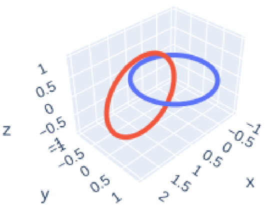
Popular nonlinear approach: seek **autoencoder** $G \circ F$, where the output of the **encoder** $F\colon \mathbb{R}^n \to \mathbb{R}^k$ is the desired k parameters, $G\colon \mathbb{R}^k \to \mathbb{R}^n$ is the **decoder**, and F, G are continuous.

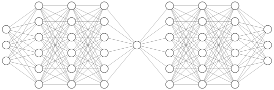
Often F, G are artificial neural network functions.

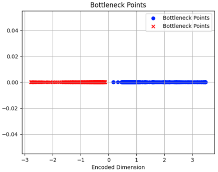
Ideal autoencoder: $G(F(x)) = x$ for all $x \in K$.

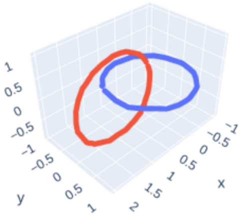
These **do not usually exist!** Existence $\implies$ K is homeomorphic to a subset of $\mathbb{R}^k$, which is not true of most k-dimensional K.

If autoencoding should not work, how does it?[2] Example:









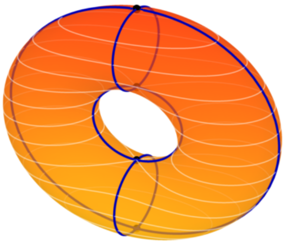
$K =$ a pair of circles in $\mathbb{R}^3$, after thickening then deleting small intervals, is diffeomorphic to a pair of disjoint intervals in $\mathbb{R}$.

Encoder $F\colon \mathbb{R}^3 \to \mathbb{R} =$ any extension of this diffeomorphism.
Decoder $G\colon \mathbb{R} \to \mathbb{R}^3 =$ any extension of inverse diffeomorphism.

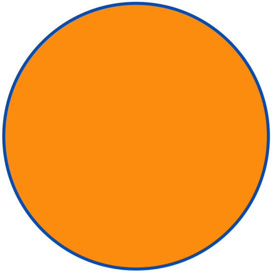
Such small intervals disjoint from the data set always exist.

[2] MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

If autoencoding should not work, how does it? In general:



$\longleftrightarrow$



K = a union of $\leq k$-dimensional compact submanifolds of $\mathbb{R}^n$, after thickening then deleting the codimension > 0 "steepest ascent disks" of a polar Morse function,[3] is diffeomorphic to a subset of $\mathbb{R}^k$.

Encoder $F\colon \mathbb{R}^n \to \mathbb{R}^k$ = any extension of this diffeomorphism.
Decoder $G\colon \mathbb{R}^k \to \mathbb{R}^n$ = any extension of inverse diffeomorphism.

Can always find such a "codim > 0 set" disjoint from the data.

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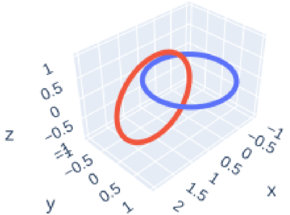
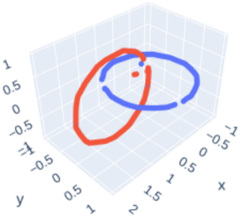
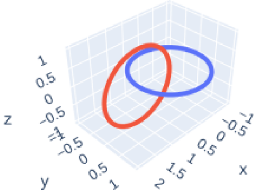
[3] A *navigation function*, in the parlance of Rimon and Koditschek (1990).

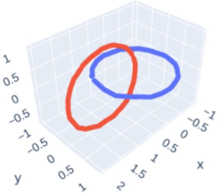
Note: training sometimes yields disconnected “good” sets





In practice, random initialization/training leads to random outcomes, even those with disconnected “good sets”, despite the fact that arbitrarily large connected “good sets” (disks) exist.





Semi-global autoencoders always exist[4]

Let $\mathcal{F}^{\ell,m}$ be dense in the space of continuous functions $\mathbb{R}^\ell \to \mathbb{R}^m$, e.g., the collection of possible neural network outputs.

Theorem 1 (MDK and E D Sontag). Let $K \subset \mathbb{R}^n$ be finitely many disjoint compact $\leq k$-dimensional submanifolds with(out) boundary, and let μ, $\partial\mu$ be any smooth measures on K, ∂K. For each $\delta > 0$ and finite set $S \subset K$, there is a closed set $K_0 \subset K$ s.t.

- $K_0 \cap S = \varnothing$, $\mu(K_0) < \delta$, $\partial\mu(K_0 \cap \partial K) < \delta$;
- $M \setminus K_0$ is connected for each component M of K;
- For each $\varepsilon > 0$ there are functions $F \in \mathcal{F}^{n,k}$, $G \in \mathcal{F}^{k,n}$ s.t.

$$\sup_{x \in K \setminus K_0} \|G(F(x)) - x\| < \varepsilon.$$

$\implies$ data S can be reconstructed to order ε, and generalization error will also be uniformly smaller than ε with probability $1 - \delta$.

[4] MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

Almost-global autoencoders do not generally exist

Theorem 2 (MDK and EDS). Let $K \subset \mathbb{R}^n$ be a k-dimensional compact submanifold without boundary. For any continuous functions $F\colon \mathbb{R}^n \to \mathbb{R}^k$ and $G\colon \mathbb{R}^k \to \mathbb{R}^n$,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\textbf{reach}} > 0.$$

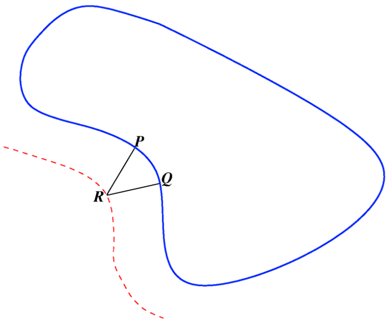


Figure: The **reach** $r_K > 0$ of K is the largest number such that any $x \in \mathbb{R}^n$ satisfying $\operatorname{dist}(x, K) < r_K$ has a unique nearest point on K. Both line segments shown have length r_K.

Almost-global autoencoders do not generally exist[5]

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$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r_K}_{\textbf{reach}} > 0. \tag{2}$$

Proof:

- $N := \{x \in \mathbb{R}^n\colon \mathrm{dist}(x, K) < r_K\}$ contains line segment from $x \in N$ to nearest $\rho(x) \in N$; ρ is continuous.

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- If (2) does not hold, $t \mapsto \rho \circ (tG \circ F|_K + (1-t)\,\mathrm{id}_K)$ is a homotopy of id_K to $\rho \circ G \circ F|_K$, so $\deg_2(\rho \circ G \circ F|_K) = 1$.

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- But this contradicts

$$0 = \deg_2(\rho \circ G \circ F|_K) = \deg_2(\rho \circ G|_{F(K)}) \underbrace{\deg_2(F|_K)}_{0}.$$

[5] MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

Example: $K = 2$ unit circles; max error $>$ reach $r_K = 1$

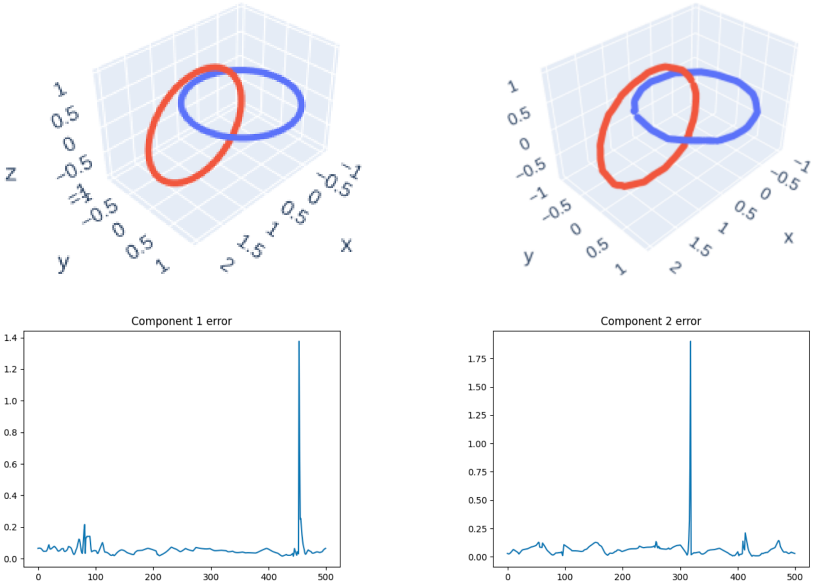
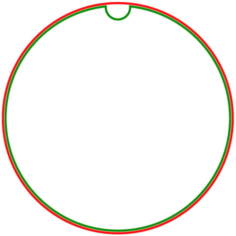


Figure: Errors $\|G(F(x)) - x\|$ on the two circles. The x-axis shows the index k representing the k-th evenly-spaced point on the respective circle.

In fact, true min-max error is usually bigger than the reach[6]



Red reach $= 1$ but green reach $= \varepsilon \ll 1$, so previous min max error "reach" is conservative. But green "**dewrinkled reach**" $= 1 - \varepsilon$.

Corollary (MDK and EDS). Let $K \subset \mathbb{R}^n$ be a k-dimensional compact submanifold without boundary. For any continuous functions $F \colon \mathbb{R}^n \to \mathbb{R}^k$ and $G \colon \mathbb{R}^k \to \mathbb{R}^n$,

$$\sup_{x \in K} \|G(F(x)) - x\| \geq \underbrace{r^*_{K,k}}_{\textbf{dewrinkled reach}} := \sup_{L \in \mathcal{M}_{n,k}, T \in C(L \to K)} \{r_L - \delta(T)\}.$$

[6] MDK and E D Sontag. *Why should autoencoders work?* Transactions on Machine Learning Research (2024).

Implications for autoencoder training error[7]

Theorem 1 $\implies$ F**,** G **always exist making the** $L^2(\mu)$ **loss**

$$\int_K \|G(F(x)) - x\|^2 d\mu(x) < \varepsilon$$

$$\left(= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} G(F(x_i)) - x_i\|^2 \right)$$

arbitrarily small (any G can be modified off of $F(K \setminus K_0)$ to make the autoencoder error smaller than a certain $C_K > 0$ on K_0.)

However, Theorem 2 $\implies$ **for many** K**, the** L^∞ **loss**

$$\max \|G(F(x)) - x\| \geq r_K > 0$$

is uniformly big, independent of F**,** G.

[7] We thank Dr. Joshua Batson for suggesting these observations.

Summary

Main representation result: data lying in a submanifold K of dimension k can be encoded through a bottleneck layer of the same dimension k, up to an arbitrarily small reconstruction error ε.

Moreover, the generalization error will also be uniformly smaller than ε with arbitrarily high probability $1 - \delta$.

Main necessity result: for many K (including all K without boundary), there is a geometric lower bound on the global reconstruction error.

Training implications: L^2 error can always be made arbitrarily small; L^∞ error cannot.

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Feedback stabilizability

Identifying engineering (im)possibilities for:

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- Many assume the dynamical system is globally linearizable
- Which ones are?
- 1-parameter subgroups of torus actions with asymptotic phase

Feedback stabilizability

“Applied Koopmanism”

“*A central focus of modern Koopman analysis is to find a finite set of nonlinear measurement functions, or coordinate transformations, in which the dynamics appear linear.*”

— Brunton, Budišić, Kaiser, and Kutz. “Modern Koopman Theory for Dynamical Systems.” SIAM Review, 64.2 (2022)

They seek nonlinear measurements that separate points, to be able to invert / not to lose information. Also want measurements / inverse to be continuous for practical reasons.

More formally, they seek *embeddings* of *nonlinear* dynamical systems into *linear* ones as invariant subsets, so that existing theoretical and algorithmic linear tools can be utilized.

Linearizing embeddings

Let f be a locally Lipschitz vector field on a manifold M. Consider

$$\dot{x} = \frac{d}{dt}x = f(x),$$

assume this ODE's solutions $x(t) = \Phi^t(x_0)$ are defined for all time.

$F\colon M \to \mathbb{R}^n$ is a **topological embedding** if F is a one-to-one continuous map with a continuous inverse $F^{-1}\colon F(M) \to M$, and is a **smooth embedding** if additionally F, F^{-1} are smooth.

Such an embedding F is **linearizing** if $F \circ \Phi^t = e^{Bt} \circ F$ for some $n \times n$ matrix B. In the smooth case, $y = F(x)$ satisfies $\dot{y} = By$.

Fundamental question: when is (M, Φ) linearizable in this sense?

When is a dynamical system (M, Φ) globally linearizable?

- Not when M is connected, forward Φ-trajectories are precompact, and Φ has a countable number ≥ 2 of omega limit sets (Liu, Ozay, Sontag 2023).
- Not when M is connected and Φ has a non-global compact attractor $A \neq \varnothing$, since its open basin of attraction would also be closed (by the Jordan normal form theorem), hence empty.

Thus, we study linearizability of the restriction (S, Φ) of Φ to

1. compact invariant sets S, and
2. basins S of compact attractors A.

For these 2 cases we obtain **necessary and sufficient conditions for global linearizability** of (S, Φ) by an embedding, for the 2 cases of topological and smooth embeddings (4 cases total).[8]

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[8] MDK and P. Arathoon, *Linearizability of flows by embeddings (2023).*

Torus preliminaries

The n-torus $T = T^n$ is (Lie group) isomorphic to $(\mathbb{R}/\mathbb{Z})^n$, vectors with n real entries but with addition defined elementwise modulo 1.

A **torus action** on S is a map $\Theta\colon T \times S \to S$ satisfying $\Theta^{\tau_1+\tau_2}(s) = \Theta^{\tau_1} \circ \Theta^{\tau_2}(s)$ for all $s \in S$ and $\tau_1, \tau_2 \in T$.

The flow (S, Φ) is a **1-parameter subgroup of a torus action** if $\Phi^t = \Theta^{\omega t \text{ mod } 1}$ for some torus action Θ on S, $\omega \in \mathbb{R}^n$.

The linearizability theorem, case 1: compact, smooth

Observation: If (S, Φ) is linearizable with S compact, the Jordan normal form theorem implies (S, Φ) embeds into the flow on $\mathbb{C}^n$ of a diagonal imaginary matrix, so (S, Φ) is a 1-parameter subgroup of restriction of standard torus action of T^n on $\mathbb{C}^n$ to a subtorus.

This gives one implication below; the *Mostow-Palais equivariant embedding theorem* gives the other.

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Theorem (MDK and P. Arathoon). If S is a compact embedded submanifold, (S, Φ) is linearizable by a smooth embedding $\iff$ (S, Φ) is a 1-parameter subgroup of a smooth torus action.

We use this theorem to construct examples of smoothly linearizable (S, Φ) having isolated equilibria with e.g. $S =$ a sphere, torus, Klein bottle. On the other hand, regarding *non*linearizability...

Topological implications for case 1 (compact, smooth)

If (S, Φ) is a 1-parameter subgroup of a smooth torus action, Bochner's linearization theorem yields an $n \times n$ skew matrix B_e and a system of local coordinates on a neighborhood of each equilibrium $e \in S$ such that $\Phi^t \approx e^{B_e t}$. Hence if e is isolated then B_e is invertible, $n = \dim S$ is even, and the Hopf index of e is $+1$.

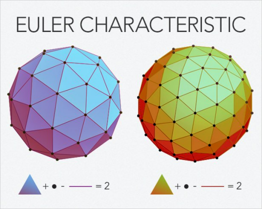
Corollary (MDK and PA). If S is an odd-dimensional connected compact submanifold with at least one isolated equilibrium, then (S, Φ) cannot be linearized by a smooth embedding.

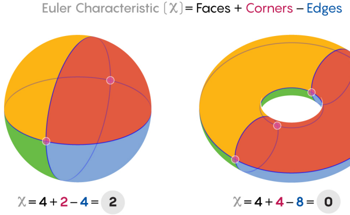
Corollary (MDK and PA).[9] If S is a compact submanifold containing at most finitely many equilibria such that (S, Φ) is linearizable by a smooth embedding, $\underbrace{\chi(S)}_{\text{Euler char.}} = \#\{\text{equilibria}\} \geq 0$.

[9] Apply the Poincaré-Hopf theorem.

A primer on the Euler characteristic[10]

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).





Notation: $\chi(Y) :=$ Euler characteristic of Y.

Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\Sigma_g) = 2 - 2g$

Σ_g for $g = 1, 2, 3$ (not linearizable for $g > 1$ if finite equilibria).

[10] Figures from Quanta Magazine and Wikipedia.

The linearizability theorem, case 2: compact, continuous

The theorem for case 2 is similar for case 1, but an additional assumption is needed to rule out a pathology not possible in case 1.

Theorem (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff$ (S, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

A torus action has **finitely many orbit types** if there are only finitely many subgroups $H \subset T$ such that $H = \{\tau \in T : \Theta^\tau(s) = s\}$ is the fixed point set of some $s \in S$.

Another point of view: quasiperiodic pinched torus families

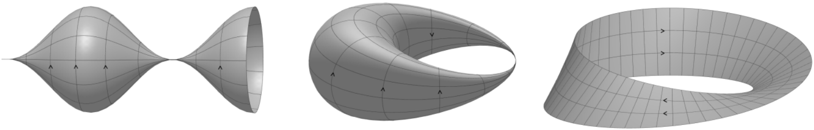


Figure: examples of quasiperiodic pinched torus families

Proposition (MDK and PA). If S is compact, (S, Φ) is linearizable by a topological embedding $\iff$ (S, Φ) is a quasiperiodic pinched torus family.

Definition. P is a **pinched torus family** if there are $m, n \in \mathbb{N}$, closed subsets $C_1, \ldots, C_n \subset B \subset T^m$, and a continuous group homomorphism $F\colon T^n \to T^m$ such that P is the quotient of $F^{-1}(B)$ by collapsing the j-th $(\mathbb{R}/\mathbb{Z})$-factor of $F^{-1}(C_j) \subset T^n$ for all j. A pinched torus family P is **quasiperiodic** if it is equipped with the induced flow generated by any $\omega \in \mathbb{R}^n$ with $\mathrm{T}F(\omega) = 0$.

The linearizability theorem, case 3: basin, continuous

If S is the basin of an asymptotically stable compact set $A \subset S$, A has continuous (smooth) **asymptotic phase**[11] if there is a continuous (smooth) **asymptotic phase map** $P\colon S \to A$, i.e.,

$$P|_A = \mathrm{id}_A, \qquad P \circ \Phi^t|_S = \Phi^t \circ P \quad \text{for all } t \in \mathbb{R}.$$

Theorem (MDK and PA). (S, Φ) is linearizable by a topological embedding $\iff$ A has continuous asymptotic phase and (A, Φ) is a 1-parameter subgroup of a continuous torus action with finitely many orbit types.

Example. The basin of an asymptotically stable limit cycle is linearizable by a topological embedding $\iff$ the cycle has continuous asymptotic phase. This is not always the case, but it is the case if $\Phi \in C^1$ and the cycle is hyperbolic.

[11] This notion has roots in oscillator theory and more generally NHIM theory.

The linearizability theorem, case 4: basin, smooth

Theorem (MDK and PA). (S, Φ) is linearizable by a smooth embedding $\iff$ A is an embedded submanifold with smooth asymptotic phase, (A, Φ) is a 1-parameter subgroup of a smooth torus action, and for some open $U \supset A$, (U, Φ) embeds in a reducible linear flow covering Φ on some vector bundle over A.

When does the final condition hold? Classical linearization theorems and recent linearizing semiconjugacy theorems (MDK and Revzen, 2023) give answers in the special cases that A is an equilibrium or periodic orbit, and some things are known if A is quasiperiodic, but the general case seems to be an open problem.

A necessary condition for A to satisfy all conditions of the theorem is that A be a (eventually relatively ∞-)**normally hyperbolic invariant manifold**. See Eldering, MDK, Revzen (2018) for related results on asymptotic phase and linearizability.

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- Brockett's necessary condition and beyond
- A homotopy theorem beyond the Coron/Mansouri tests
- Periodic orbits can be easier to stabilize than equilibria

Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \tag{1}$$

where $M \ni x$ is a smooth manifold and f is smooth.

1. **Controllability problem**: Given $a, b \in M$, find $u(t)$ s.t. $x(T) = b$ if $x(0) = a$ for some $T > 0$.

$$a \rightsquigarrow b$$

2. **Stabilizability problem**: Given a compact subset $A \subset M$, find smooth $u(x)$ s.t. A is **asymptotically stable**[12] for the **closed-loop vector field** $F(x) = f(x, u(x))$. ▸ Link

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[12] For every open $W \supset A$ there is an open $V \supset A$ s.t. all forward F-trajectories initialized in V are contained in W and converge to A.

The stabilization conjecture and Brockett's solution

Often $A = \{x_*\}$ is a point, $M = \mathbb{R}^n$ in the stabilization problem.

Stabilization conjecture (pre-1983): a reasonably strong form of controllability implies smooth stabilizability of a point.

Example: the "Heisenberg system" or "nonholonomic integrator"

$$\left.\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned}\right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?

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Theorem (Brockett). If a point is stabilizable, then image(f) is a neighborhood of 0. (In the example, $(0, 0, \varepsilon) \notin$ image(f).)

Other stabilizability work

- Exponential (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
- global (Byrnes 2008, Baryshnikov 2023),
- time-varying (Coron 1992), and
- discontinuous (Clarke, Ledyaev, Sontag, Subbotin 1997)

variants of the stabilization problem are not considered here.

Coron's and Mansouri's obstructions

Krasnosel'skiĭ and Zabreĭko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \colon f(x, u) \neq 0\}.$$

Theorem (Coron 1990). If $n > 1$ and a point is stabilizable,

$$f_*(H_{n-1}(\Sigma)) = H_{n-1}(\mathbb{R}^n \setminus \{0\}) \qquad (\cong \mathbb{Z}).$$

Theorem (Mansouri 2010). If a closed codimension > 1 submanifold $A \subset \mathbb{R}^n$ with Euler characteristic $\chi(A)$ is stabilizable,

$$f_*(H_{n-1}(\Sigma)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \qquad (\cong \chi(A) \cdot \mathbb{Z}).$$

Limitations of these results

The results of Brockett, Coron, Mansouri rely on parallelizability of $\mathbb{R}^n$ to view vector fields and control systems as $\mathbb{R}^n$-valued.

Furthermore, they apply only to the special case that A is a point or a closed submanifold of $\mathbb{R}^n$ with $\chi(A) \neq 0$.

But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

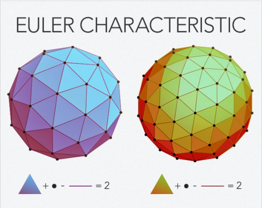
How to test for stabilizability in such general settings?[13]

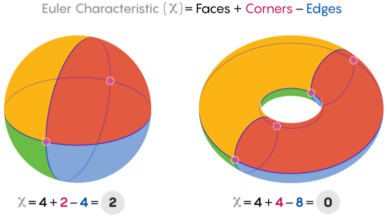
- **Generalization of Brockett's test** (MDK and Daniel E. Koditschek, J Geometric Mechanics, 2022).
- **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).

[13] An exposition of all stabilizability results here is in 2023 book *Topological Obstructions to Stability and Stabilization* by W. Jongeneel and E. Moulay.

A primer on the Euler characteristic[14]

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).





Notation: $\chi(Y) :=$ Euler characteristic of Y.

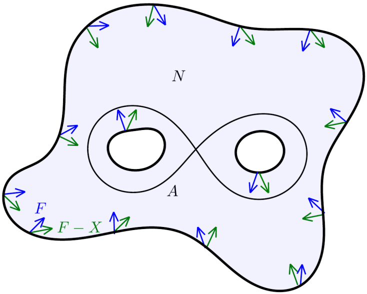
Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\text{figure 8}) = -1$

Theorem (Poincaré, Hopf): if N is a compact smooth manifold with boundary ∂N, then $\chi(N) = 0 \iff$ there exists a nowhere-zero smooth vector field on N pointing inward at ∂N.

[14] Figures from Quanta Magazine.

Generalization of Brockett's test

Theorem (MDK & Koditschek 2022): Let $A \subset M$ be compact & stabilizable. Then $\chi(A)$ is well-defined. If $\chi(A) \neq 0$, then for any sufficiently small vector field X, $X(x_0) = f(x_0, u_0)$ for some x_0, u_0.



Proof: Assume $\exists$ stabilizing $u(x)$ and define $F(x) := f(x, u(x))$. Lyapunov function theory $\implies \exists$ compact smooth domain $N \supset A$ s.t. F points inward at ∂N and $\chi(A) = \chi(N) \neq 0$. Continuity $\implies F - X$ points inward at ∂N if X is small $\implies F - X$ has a zero by Poincaré-Hopf $\implies \exists x_0$ s.t. $X(x_0) = F(x_0) = f(x_0, u(x_0))$.

Examples

Heisenberg system

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= v\\ \dot{z} &= yu - xv\end{aligned} \tag{2}$$

Kinematic differential drive robot

$$\begin{aligned}\dot{x} &= u\cos\theta\\ \dot{y} &= u\sin\theta\\ \dot{\theta} &= v\end{aligned} \tag{3}$$

The right side of (2) $\neq X_\varepsilon := (0,0,\varepsilon)$ for any $\varepsilon > 0$.

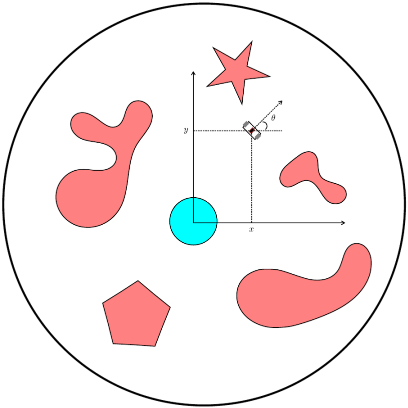
The right side of (3) $\neq X_\varepsilon := (\varepsilon\sin\theta, -\varepsilon\cos\theta, 0)$ for any $\varepsilon > 0$.

Thus, our result $\implies$ A is not stabilizable if $\chi(A) \neq 0$. E.g., if A is a stabilizable compact submanifold, A is a union of circles and tori.

Other applications: any stabilizable compact set has zero Euler characteristic for satellite orientation with ≤ 2 thrusters, for nonholonomic dynamics with ≥ 1 global constraint 1-form,...

Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some “bad” set immediately enter some “good” set.



E.g., impossible for this differential drive robot to aim within ± 179 degrees of the origin while “strictly” avoiding obstacles via $u(x)$.

Homotopy theorem & generalized Coron, Mansouri tests

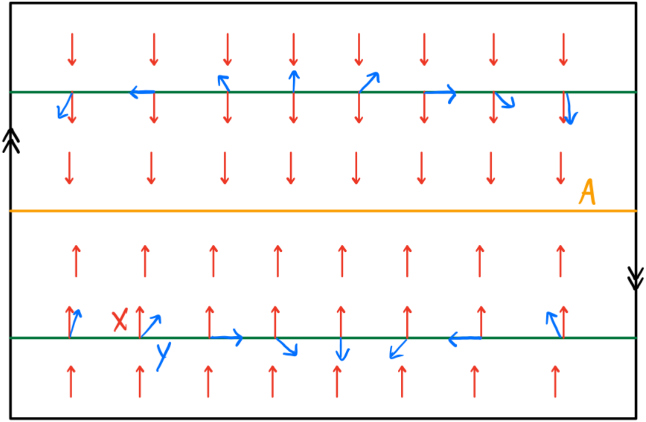
Homotopy theorem (MDK 2023). Let X, Y be smooth vector fields on a manifold M with a compact set $A \subset M$ asymptotically stable for both. There is an open set $U \supset A$ such that $X|_{U\setminus A}$, $Y|_{U\setminus A}$ are homotopic through nowhere-zero vector fields.

$\implies$ **Theorem (MDK 2023).** Let the compact set $A \subset M$ be asymptotically stable for *some* smooth vector field Y on M. If A is stabilizable for $\dot{x} = f(x, u)$, then for all small enough open $U \supset A$,

$$H_\bullet(T(U \setminus A) \setminus 0) \supset \underbrace{f_* H_\bullet(\Sigma) \supset Y_* H_\bullet(U \setminus A)}_{\text{cf. Coron, Mansouri}}.$$

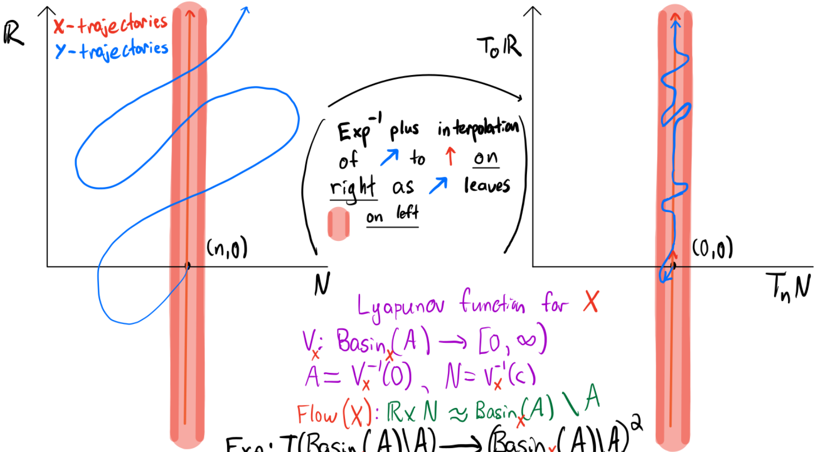
These are stronger than all preceding results: there is an example (MDK 2023) for which non-stabilizability is detected by each of these theorems but not by any of the preceding theorems.

Möbius strip example



X $\not\simeq$ y since y ⟳ twice around ◯ w.r.t. X while X ⟳ zero times w.r.t. X $\Rightarrow$ A is not asymptotically stable for y by the homotopy theorem.

Proof of the homotopy theorem



Lyapunov function for X

$V_X: \mathrm{Basin}_X(A) \to [0,\infty)$

$A = V_X^{-1}(0)$, $N = V_X^{-1}(c)$

$\mathrm{Flow}(X): \mathbb{R}\times N \approx \mathrm{Basin}_X(A)\setminus A$

$\mathrm{Exp}: T(\mathrm{Basin}_X(A)\setminus A) \longrightarrow (\mathrm{Basin}_X(A)\setminus A)^2$

w.r.t. product of any metric on N with Euclidean metric on $\mathbb{R}$

Can these results detect stabilizability of periodic orbits?

If A is the image of a periodic orbit with the same orientation for X and Y, the straight-line homotopy over a sufficiently small open $U \supset A$ satisfies the homotopy theorem's conclusion regardless of whether A is attracting, repelling, or neither for X or Y.

$\Longrightarrow$ **homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, **preceding results also give no information.**

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If A is the image of a periodic orbit with the same orientation for X and Y, the straight-line homotopy over a sufficiently small open $U \supset A$ satisfies the homotopy theorem's conclusion regardless of whether A is attracting, repelling, or neither for X or Y.

$\implies$ **homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, **preceding results also give no information.**

...Could it be that periodic orbits might be "easy" to stabilize?

Periodic orbits are sometimes easier to stabilize

Indeed, at least sometimes:

Theorem (Anthony M. Bloch & MDK, in preparation).
For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!

Identifying engineering (im)possibilities for:

Deep neural network autoencoders

Applied Koopman operator methods

Feedback stabilizability

- Brockett's necessary condition and beyond
- A homotopy theorem beyond the Coron/Mansouri tests
- Periodic orbits can be easier to stabilize than equilibria

Thank you for your attention.

Identifying engineering (im)possibilities for:

Deep neural network autoencoders

- They should not work, and yet they do: resolving the paradox
- Training implications: L^2 but not L^∞ error can be made small

Applied Koopman operator methods

- Many assume the dynamical system is globally linearizable
- Which ones are?
- 1-parameter subgroups of torus actions with asymptotic phase

Feedback stabilizability

- Brockett's necessary condition and beyond
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- Periodic orbits can be easier to stabilize than equilibria